

# POISSON 2-GROUPS

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## Abstract

We prove a 2-categorical analogue of a classical result of Drinfeld: there is a one-to-one correspondence between connected, simply-connected Poisson Lie 2-groups and Lie 2-bialgebras. In fact, we also prove that there is a one-to-one correspondence between connected, simply connected quasi-Poisson 2-groups and quasi-Lie 2-bialgebras. Our approach relies on a “universal lifting theorem” for Lie 2-groups: an isomorphism between the graded Lie algebras of multiplicative polyvector fields on the Lie 2-group on one hand and of polydifferentials on the corresponding Lie 2-algebra on the other hand.

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## Introduction

A Poisson group is a Lie group equipped with a compatible Poisson structure. Poisson groups are the classical limit of quantum groups and have been extensively studied in the past two decades. For instance, Drinfeld proved that there is a bijection between connected, simply connected Poisson groups and Lie bialgebras [7, 8].

Lie 2-groups (also called strict Lie 2-groups in the literature) are Lie group objects in the category of Lie groupoids, or equivalently Lie groupoid objects in the category of Lie groups. More explicitly, a Lie 2-group is a Lie groupoid  $\Gamma_1 \rightrightarrows \Gamma_0$ , where both  $\Gamma_1$  and  $\Gamma_0$  are Lie groups and all the groupoid structure maps are group homomorphisms. Lie 2-groups are special instances of Mackenzie's double groupoids [14].

The recent categorification trend motivates the search for an appropriate notion of quantum 2-groups. Poisson 2-groups are a natural first step in that direction. By a Poisson 2-group, we mean a Lie 2-group equipped with a Poisson structure  $\Pi$  on  $\Gamma_1$ , which is multiplicative with respect to both the group and the groupoid structures on  $\Gamma_1$ . In other words,  $(\Gamma_1, \Pi)$  is simultaneously both a Poisson group [7, 8] and a Poisson groupoid [20].

Lie 2-algebras are Lie algebra objects in the category of Lie algebroids [1]. They can be identified with Lie algebra crossed modules: pairs of Lie algebras  $\theta$  and  $\mathfrak{g}$  together with a linear map  $\phi : \theta \rightarrow \mathfrak{g}$  and an action of  $\mathfrak{g}$  on  $\theta$  by derivations satisfying a certain compatibility condition. Likewise, a Lie 2-bialgebra can be considered as a Lie bialgebra crossed module, i.e. a pair of Lie algebra crossed modules in duality:  $(\theta \xrightarrow{\phi} \mathfrak{g})$  and  $(\mathfrak{g}^* \xrightarrow{-\phi^*} \theta^*)$  are both Lie algebra crossed modules, and  $(\mathfrak{g} \ltimes \theta, \theta^* \ltimes \mathfrak{g}^*)$  is a Lie bialgebra.

We prove that, at the infinitesimal level, Poisson 2-groups induce Lie 2-bialgebras. More precisely, we prove the following analogue of Drinfeld's theorem:

**Theorem A.** *There is a one-to-one correspondence between connected, simply-connected Poisson 2-groups and Lie 2-bialgebras.*

We will also prove a more general result:

**Theorem B.** *There is a one-to-one correspondence between connected, simply-connected quasi-Poisson 2-groups and quasi-Lie 2-bialgebras.*

Quasi-Poisson 2-groups are, in a certain sense, the 2-categorical analogues of Kosmann-Schwarzbach’s quasi-Poisson groups [10]. A quasi-Poisson 2-group is a Lie 2-group  $\Gamma_1 \rightrightarrows \Gamma_0$  endowed with a multiplicative quasi-Poisson structure on  $\Gamma_1$ , i.e. a multiplicative bivector field  $\Pi$  on  $\Gamma_1$  such that the Schouten bracket  $[\Pi, \Pi]$  is some sort of coboundary.

A natural generalization of Lie 2-algebras (or Lie algebra crossed modules), weak Lie 2-algebras are two-term  $L_\infty$  algebras. They can be described concisely in terms of the shifted degree “big bracket,” which is a Gerstenhaber bracket on  $S^\bullet(V[2] \oplus V^*[1])$ . Here  $V = \theta \oplus \mathfrak{g}$  is a graded vector space, where  $\theta$  is of degree 1 and  $\mathfrak{g}$  is of degree 0. Identifying  $S^\bullet(V[2] \oplus V^*[1])$  with the space  $\Gamma(\wedge^\bullet T[4]M)$  of polyvector fields on  $M = V^*[-2]$  with polynomial coefficients, the big bracket can be simply described as the Schouten bracket of polyvector fields on  $M$ .

In [6], we developed a notion of weak Lie 2-bialgebras: objects that are simultaneously weak Lie 2-algebras as well as weak Lie 2-coalgebras, both structures being compatible with one another in a certain sense. In terms of the big bracket, a weak Lie 2-bialgebra on a graded vector space  $V$  is a degree-(-4) element  $t$  of  $S^\bullet(V[2] \oplus V^*[1])$  satisfying  $\{t, t\} = 0$ . Quasi-Lie 2-bialgebras are a special instance of weak Lie 2-bialgebras.

Our proofs of Theorems A and B rely on the following “universal lifting theorem,” which should be of independent interest:

**Theorem C.** *Given a Lie 2-group  $\Gamma_1 \rightrightarrows \Gamma_0$ , if both  $\Gamma_1$  and  $\Gamma_0$  are connected and simply connected, then the graded Lie algebras  $\bigoplus_{k \geq 0} \mathfrak{X}_{\text{mult}}^k(\Gamma_1)$  and  $\bigoplus_{k \geq 0} \mathcal{A}_k$  are isomorphic.*

Here  $\bigoplus_{k \geq 0} \mathfrak{X}_{\text{mult}}^k(\Gamma_1)$  denotes the space of multiplicative polyvector fields on  $\Gamma_1$  which, being closed with respect to the Schouten bracket, is naturally a graded Lie algebra. On the other hand,  $\bigoplus_{k \geq 0} \mathcal{A}_k$  denotes the graded Lie algebra formed by the polydifferentials on the associated Lie 2-algebra — the infinitesimal counterparts of the multiplicative polyvector fields on the Lie 2-group.

Theorems A and B are proved simply by expressing the algebraic data defining the weak Lie 2-bialgebra structure in terms of the graded Lie algebra  $\bigoplus_{k \geq 0} \mathcal{A}_k$ .

We refer to the recent papers [13, 16, 18] on integration of Courant algebroids to symplectic 2-groupoids, which may have a close connection to our work.

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## 1. Quasi-Lie 2-bialgebras

In this section, we recall some basic notions regarding quasi Lie 2-bialgebras developed in [6].

**1.1. The big bracket.** We will introduce a graded version of the big bracket [11, 12] involving graded vector spaces.

Let  $V = \bigoplus_{k \in \mathbb{Z}} V^{(k)}$  be a graded vector space. Consider the  $\mathbb{Z}$ -graded manifold  $M = V^*[-2]$  and the shifted tangent space

$$T[4]M \cong (M \times V^*[-2])[4] \cong M \times V^*[2].$$

Consider the space of polyvector fields on  $M$  with polynomial coefficients:

$$\begin{aligned} \Gamma(\wedge^\bullet T[4]M) &\cong S^\bullet(M^*) \otimes S^\bullet((V^*[2])[-1]) \\ &\cong S^\bullet(V[2]) \otimes S^\bullet(V^*[1]) \cong S^\bullet(V[2] \oplus V^*[1]). \end{aligned}$$

We write  $\mathcal{S}^\bullet$  for  $S^\bullet(V[2] \oplus V^*[1])$  and  $\odot$  for the symmetric tensor product in  $\mathcal{S}^\bullet$ .

There is a standard way to endow  $\mathcal{S}^\bullet = \Gamma(\wedge^\bullet T[4]M)$  with a graded Lie bracket, i.e. the Schouten bracket, which is denoted by  $\{\cdot, \cdot\}$ . It is a bilinear map  $\{\cdot, \cdot\} : \mathcal{S}^\bullet \otimes \mathcal{S}^\bullet \rightarrow \mathcal{S}^\bullet$  satisfying the following properties:

- 1)  $\{v, v'\} = \{\epsilon, \epsilon'\} = 0$ , for all  $v, v' \in V[2]$  and  $\epsilon, \epsilon' \in V^*[1]$ ;
- 2)  $\{v, \epsilon\} = (-1)^{|v|} \langle v | \epsilon \rangle$ , for all  $v \in V[2]$  and  $\epsilon \in V^*[1]$ ;
- 3)  $\{e_1, e_2\} = -(-1)^{(|e_1|+3)(|e_2|+3)} \{e_2, e_1\}$ , for all  $e_1, e_2 \in \mathcal{S}^\bullet$ ;
- 4)  $\{e_1, e_2 \odot e_3\} = \{e_1, e_2\} \odot e_3 + (-1)^{(|e_1|+3)|e_2|} e_2 \odot \{e_1, e_3\}$ , for all  $e_1, e_2, e_3 \in \mathcal{S}^\bullet$ .

It is clear that  $\{\cdot, \cdot\}$  is of degree 3, i.e.

$$|\{e_1, e_2\}| = |e_1| + |e_2| + 3,$$

for all homogeneous  $e_i \in \mathcal{S}^\bullet$ , and the following graded Jacobi identity holds:

$$\{e_1, \{e_2, e_3\}\} = \{\{e_1, e_2\}, e_3\} + (-1)^{(|e_1|+3)(|e_2|+3)} \{e_2, \{e_1, e_3\}\}.$$

Hence  $(\mathcal{S}^\bullet, \odot, \{\cdot, \cdot\})$  is a Schouten algebra, also known as an odd Poisson algebra, or a Gerstenhaber algebra [19].

Due to our degree convention, when  $V$  is an ordinary vector space considered as a graded vector space concentrated at degree 0, the bracket above is different from the usual big bracket in the literature [11].

**1.2. Quasi-Lie 2-bialgebras.** Following Baez-Crans [1], a weak Lie 2-algebra is an  $L_\infty$ -algebra on the 2-term graded vector space  $V = \theta \oplus \mathfrak{g}$ , where  $\theta$  is of degree 1 and  $\mathfrak{g}$  is of degree 0. Unfolding the  $L_\infty$ -structure, we can define a weak Lie 2-algebra as a pair of vector spaces  $\theta$  and  $\mathfrak{g}$  endowed with the following structures:

- 1) a linear map  $\phi: \theta \rightarrow \mathfrak{g}$ ;
- 2) a bilinear skewsymmetric map  $[\cdot, \cdot]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ ;
- 3) a bilinear map  $\cdot \triangleright \cdot: \mathfrak{g} \otimes \theta \rightarrow \theta$ ;
- 4) a trilinear skewsymmetric map  $h: \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \rightarrow \theta$ , called the homotopy map.

These maps are required to satisfy the following compatibility conditions: for all  $w, x, y, z \in \mathfrak{g}$  and  $u, v \in \theta$ ,

$$\begin{aligned} [[x, y], z] + [[y, z], x] + [[z, x], y] + (\phi \circ h)(x, y, z) &= 0; \\ y \triangleright (x \triangleright u) - x \triangleright (y \triangleright u) + [x, y] \triangleright u + h(\phi(u), x, y) &= 0; \\ \phi(u) \triangleright v + \phi(v) \triangleright u &= 0; \\ \phi(x \triangleright u) &= [x, \phi(u)]; \end{aligned}$$

and

$$\begin{aligned} -w \triangleright h(x, y, z) - y \triangleright h(x, z, w) + z \triangleright h(x, y, w) + x \triangleright h(y, z, w) \\ = h([x, y], z, w) - h([x, z], y, w) + h([x, w], y, z) \\ + h([y, z], x, w) - h([y, w], x, z) + h([z, w], x, y). \end{aligned}$$

If  $h$  vanishes, we call it a strict Lie 2-algebra, or simply a Lie 2-algebra.

Now consider the degree shifted vector spaces  $V[2]$  and  $V^*[1]$ . Under such a degree convention, the degrees of  $\mathfrak{g}$ ,  $\theta$ ,  $\mathfrak{g}^*$  and  $\theta^*$  are specified as follows:

| space  | $\mathfrak{g}$ | $\theta$ | $\mathfrak{g}^*$ | $\theta^*$ |
|--------|----------------|----------|------------------|------------|
| degree | -2             | -1       | -1               | -2         |

We will maintain this convention throughout this section. We remind the reader that the abbreviation  $S^\bullet$  stands for  $S^\bullet(V^*[1] \oplus V[2])$ .

**Proposition 1.1** ([6]). *Under the above degree convention, a weak Lie 2-algebra structure is equivalent to a solution to the equation*

$$(1) \quad \{s, s\} = 0,$$

where  $s = \check{\phi} + \check{b} + \check{a} + \check{h}$  is an element in  $\mathcal{S}^{(-4)}$  such that

$$(2) \quad \begin{cases} \check{\phi} \in \theta^* \odot \mathfrak{g}, \\ \check{b} \in (\odot^2 \mathfrak{g}^*) \odot \mathfrak{g}, \\ \check{a} \in \mathfrak{g}^* \odot \theta^* \odot \theta, \\ \check{h} \in (\odot^3 \mathfrak{g}^*) \odot \theta. \end{cases}$$

Here the bracket in Eq. (1) stands for the big bracket as in Section 1.1.

In the sequel, we denote a weak Lie 2-algebra by  $(\theta \rightarrow \mathfrak{g}, s)$  in order to emphasize the map from  $\theta$  to  $\mathfrak{g}$ . Sometimes, we will omit  $s$  and denote a weak Lie 2-algebra simply by  $(\theta \rightarrow \mathfrak{g})$ . If  $(\mathfrak{g}^* \rightarrow \theta^*)$  is a weak Lie 2-algebra, then  $(\theta \rightarrow \mathfrak{g})$  is called a weak Lie 2-coalgebra. Equivalently, a weak Lie 2-coalgebra is a 2-term  $L_\infty$ -structure on  $\mathfrak{g}^* \oplus \theta^*$ , where  $\mathfrak{g}^*$  has degree 1 and  $\theta^*$  has degree 0.

Similarly, we have the following

**Proposition 1.2** ([6]). *A weak Lie 2-coalgebra is equivalent to a solution to the equation*

$$(3) \quad \begin{cases} \{c, c\} = 0, \\ \check{\phi} \in \theta^* \odot \mathfrak{g}, \\ \check{\epsilon} \in \theta^* \odot (\odot^2 \theta), \\ \check{\alpha} \in \mathfrak{g}^* \odot \mathfrak{g} \odot \theta, \\ \check{\eta} \in \mathfrak{g}^* \odot (\odot^3 \theta). \end{cases}$$

We denote such a weak Lie 2-coalgebra by  $(\theta \rightarrow \mathfrak{g}, c)$ .

**Definition 1.3.** *A weak Lie 2-bialgebra consists of a pair of vector spaces  $\theta$  and  $\mathfrak{g}$  together with a solution  $t = \check{b} + \check{a} + \check{h} + \check{\phi} + \check{\epsilon} + \check{\alpha} + \check{\eta} \in \mathcal{S}^{(-4)}$  to the equation  $\{t, t\} = 0$ . Here  $\check{b}, \check{a}, \check{h}, \check{\phi}, \check{\epsilon}, \check{\alpha}, \check{\eta}$  are as in Eqs. (2) and (3). If, moreover,  $\check{h} = 0$ , it is called a quasi-Lie 2-bialgebra. If both  $\check{h}$  and  $\check{\eta}$  vanish, we say that the Lie 2-bialgebra is strict, or simply a Lie 2-bialgebra.*

**Proposition 1.4.** *Let  $(\theta, \mathfrak{g}, t)$  be a weak Lie 2-bialgebra as in Definition 1.3. Then  $(\theta \rightarrow \mathfrak{g}, l)$ , where  $l = \check{\phi} + \check{b} + \check{a} + \check{h}$ , is a weak Lie 2-algebra, while  $(\theta \rightarrow \mathfrak{g}, c)$ , where  $c = \check{\phi} + \check{\epsilon} + \check{\alpha} + \check{\eta}$ , is a weak Lie 2-coalgebra.*

**Example 1.5.** Assume that  $\mathfrak{g}$  is a semisimple Lie algebra. Let  $(\cdot, \cdot)$  be its Killing form. Then  $h(x, y, z) = \hbar(x, [y, z])$ , for all  $x, y, z \in \mathfrak{g}$ , is a Lie algebra 3-cocycle, where  $\hbar$  is a constant. Let  $\theta = \mathbb{R}$ . Then the trivial map  $\mathbb{R} \rightarrow \mathfrak{g}$  together with  $h$  becomes a weak Lie 2-algebra, called the string Lie 2-algebra [1]. More precisely, the string Lie 2-algebra is as follows:

- 1)  $\theta$  is the abelian Lie algebra  $\mathbb{R}$ ;
- 2)  $\mathfrak{g}$  is a semisimple Lie algebra;
- 3)  $\phi : \theta \rightarrow \mathfrak{g}$  is the trivial map;
- 4) the action map  $\triangleright : \mathfrak{g} \otimes \theta \rightarrow \theta$  is the trivial map;
- 5)  $h : \wedge^3 \mathfrak{g} \rightarrow \theta$  is given by the map  $\hbar(\cdot, [\cdot, \cdot])$ , where  $\hbar$  is a fixed constant.

Now fix an element  $x \in \mathfrak{g}$ . We endow  $\mathbb{R} \rightarrow \mathfrak{g}$  with a weak Lie 2-coalgebra structure as follows:

- 1)  $\mathfrak{g}^*$  is an abelian Lie algebra;
- 2)  $\theta^* \cong \mathbb{R}$  is an abelian Lie algebra;
- 3)  $\phi^* : \mathfrak{g}^* \rightarrow \theta^*$  is the trivial map;
- 4) the  $\theta^*$ -action on  $\mathfrak{g}^*$  is given by  $\mathbf{1} \triangleright \xi = \text{ad}_x^* \xi$ , for all  $\xi \in \mathfrak{g}^*$ ;
- 5)  $\tilde{\eta} : \wedge^3 \theta^* \rightarrow \mathfrak{g}^*$  is the trivial map.

One can verify directly that these relations indeed define a weak Lie 2-bialgebra.

### 1.3. Lie bialgebra crossed modules.

**Definition 1.6.** A Lie algebra crossed module consists of a pair of Lie algebras  $\theta$  and  $\mathfrak{g}$ , a linear map  $\phi : \theta \rightarrow \mathfrak{g}$ , and an action of  $\mathfrak{g}$  on  $\theta$  by derivations satisfying, for all  $x, y \in \mathfrak{g}$ ,  $u, v \in \theta$ ,

- 1)  $\phi(u) \triangleright v = [u, v]$ ;
- 2)  $\phi(x \triangleright u) = [x, \phi(u)]$ ,

where  $\triangleright$  denotes the  $\mathfrak{g}$ -action on  $\theta$ .

Note that 1 and 2 imply that  $\phi$  must be a Lie algebra homomorphism. We write  $(\theta \xrightarrow{\phi} \mathfrak{g})$  to denote a Lie algebra crossed module. The associated semidirect product Lie algebra is denoted by  $\mathfrak{g} \ltimes \theta$ .

The following proposition indicates that crossed modules of Lie algebras are in one-to-one correspondence with Lie 2-algebras. We refer the reader to [1] for details.

**Proposition 1.7.** Lie algebra crossed modules are equivalent to (strict) Lie 2-algebras.

**Definition 1.8.** A Lie bialgebra crossed module is a pair of Lie algebra crossed modules in duality:  $(\theta \xrightarrow{\phi} \mathfrak{g})$  and  $(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*)$ , where  $\phi^T = -\phi^*$ , are both Lie algebra crossed modules such that  $(\mathfrak{g} \ltimes \theta, \theta^* \ltimes \mathfrak{g}^*)$  is a Lie bialgebra.

Lie bialgebra crossed modules are symmetric as we see in the next

**Proposition 1.9.** If  $((\theta \xrightarrow{\phi} \mathfrak{g}), (\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*))$  is a Lie bialgebra crossed module, so is  $((\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*), (\theta \xrightarrow{\phi} \mathfrak{g}))$ .

The following result justifies our terminology.

**Proposition 1.10.** If  $((\theta \xrightarrow{\phi} \mathfrak{g}), (\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*))$  is a Lie bialgebra crossed module, then both pairs  $(\theta, \theta^*)$  and  $(\mathfrak{g}, \mathfrak{g}^*)$  are Lie bialgebras.

**Example 1.11.** One can construct a Lie bialgebra crossed module from an ordinary Lie bialgebra as follows. Given a Lie bialgebra  $(\theta, \theta^*)$ , consider the trivial Lie algebra crossed module  $(\theta \xrightarrow{1} \theta)$ , where the second  $\theta$  acts on the first  $\theta$  by the adjoint action. In the mean time, consider the dual Lie algebra crossed module  $(\theta^* \xrightarrow{-1} \theta^*)$ , where the second  $\theta^*$  is equipped with the opposite Lie bracket:  $-[\cdot, \cdot]_*$ , and the action of the second  $\theta^*$  on the first  $\theta^*$  is given by  $\kappa_2 \triangleright \kappa_1 = -[\kappa_2, \kappa_1]_*$ , for all  $\kappa_1, \kappa_2 \in \theta^*$ . It is simple to see that  $((\theta \xrightarrow{1} \theta), (\theta^* \xrightarrow{-1} \theta^*))$  is indeed a Lie bialgebra crossed module.

The following theorem was proved in [6].

**Theorem 1.12.** There is a bijection between Lie bialgebra crossed modules and (strict) Lie 2-bialgebras.

**Example 1.13.** Consider the Lie subalgebra  $\mathfrak{u}(n) \subset \mathfrak{gl}_n(\mathbb{C})$  of  $n \times n$  skew-Hermitian matrices. Let  $\theta \subset \mathfrak{gl}_n(\mathbb{C})$  be the Lie subalgebra consisting of upper triangular matrices whose diagonal elements are real numbers. It is standard that  $(\theta, \mathfrak{u}(n))$  is a Lie bialgebra. Indeed  $\theta \oplus \mathfrak{u}(n) \cong \mathfrak{gl}_n(\mathbb{C})$ , and both  $\theta$  and  $\mathfrak{u}(n)$  are Lagrangian subalgebras of  $\mathfrak{gl}_n(\mathbb{C})$  under the nondegenerate pairing  $\langle X|Y \rangle = \text{Im}(\text{Tr}(XY))$ , for  $X, Y \in \mathfrak{gl}_n(\mathbb{C})$ . Hence  $(\theta, \mathfrak{u}(n), \mathfrak{gl}_n(\mathbb{C}))$  is a Manin triple, and thus  $(\theta, \mathfrak{u}(n))$  forms a Lie bialgebra.

Let  $\mathfrak{g}$  denote the Lie algebra of traceless upper triangular matrices with real diagonal coefficients. It turns out that  $(\theta \xrightarrow{\phi} \mathfrak{g})$ , where  $\phi$  is the map  $A \mapsto A - \text{tr}A$ , is a Lie bialgebra crossed module.

## 2. Universal lifting theorem

**2.1. Lie 2-groups.** A Lie 2-group (also called strict Lie 2-groups in the literature) is a Lie groupoid  $\Gamma_1 \rightrightarrows \Gamma_0$ , where both  $\Gamma_1$  and  $\Gamma_0$  are



Lie groups and all the groupoid structure maps are group homomorphisms. A Lie 2-group is a special case of double groupoid in the sense of Mackenzie [14].

**Definition 2.1** ([21, 22]). *A Lie group crossed module consists of a Lie group homomorphism  $\Phi : \Theta \rightarrow G$  and an action of  $G$  on  $\Theta$  by automorphisms satisfying the following compatibility conditions:*

- 1)  $\Phi(\alpha) \triangleright \beta = \alpha \beta \alpha^{-1};$
- 2)  $\Phi(g \triangleright \beta) = g \Phi(\beta) g^{-1},$

for all  $g \in G$  and  $\alpha, \beta \in \Theta$ . Here  $g \triangleright \beta$  denotes the action of  $g \in G$  on  $\beta \in \Theta$ .

We write  $(\Theta \xrightarrow{\Phi} G)$  to denote a Lie group crossed module.

**Proposition 2.2.** *There is a bijection between Lie 2-groups and crossed modules of Lie groups.*

*Proof.* This is standard. For instance, see [2, 4, 5, 15, 17]. Here we will sketch the construction of the Lie 2-group out of a crossed module, which will be needed later on.

The Lie 2-group corresponding to a Lie group crossed module  $(\Theta \xrightarrow{\Phi} G)$  will be denoted by  $G \ltimes \Theta \rightrightarrows G$ , or simply  $G \ltimes \Theta$ , by abuse of notations. Here the group structure on  $G \ltimes \Theta$  is as follows:

- group multiplication:  $(g, \alpha) \diamond (h, \beta) = (gh, (h^{-1} \triangleright \alpha) \beta);$
- group unit:  $\mathbf{1}_\diamond = (\mathbf{1}_G, \mathbf{1}_\Theta)$ , here  $\mathbf{1}_G$  and  $\mathbf{1}_\Theta$  denote, respectively, the group unit elements of  $G$  and  $\Theta$ ;
- group inversion:  $(g, \alpha)_\diamond^{-1} = (g^{-1}, (g \triangleright \alpha)^{-1}).$

The groupoid structure on  $G \ltimes \Theta \rightrightarrows G$  is as follows:

- source and target maps:  $\mathbf{s}(g, \alpha) = g, \mathbf{t}(g, \alpha) = g \Phi(\alpha);$
- groupoid multiplication:  $(g, \alpha) \star (h, \beta) = (g, \alpha \beta),$  if  $h = g \Phi(\alpha);$
- groupoid units:  $(g, \mathbf{1}_\Theta);$
- groupoid inversion:  $(g, \alpha)_\star^{-1} = (g \Phi(\alpha), \alpha^{-1}).$

q.e.d.

In the sequel, we will use Lie 2-groups and crossed modules of Lie groups interchangeably.

**2.2. Multiplicative polyvector fields on Lie groupoids.** We recall some standard results regarding multiplicative polyvector fields on a Lie groupoid. Let  $\Gamma \rightrightarrows M$  be a Lie groupoid with source and target maps  $s$  and  $t$ , respectively. Consider the graph of the groupoid multiplication  $\Lambda = \{(p, q, pq) | t(p) = s(q)\}$ , which is a submanifold in  $\Gamma \times \Gamma \times \Gamma$ .

Recall that a  $k$ -vector field  $\Sigma \in \mathfrak{X}^k(\Gamma)$  is said to be multiplicative if  $\Lambda$  is coisotropic with respect to  $\Sigma \times \Sigma \times (-1)^{k+1}\Sigma$  [9]. In other words,

$$(\Sigma \times \Sigma \times (-1)^{k+1}\Sigma)(\xi_1, \dots, \xi_k) = 0, \quad \forall \xi_1, \dots, \xi_k \in \Lambda^\perp$$

for all  $\xi_1, \dots, \xi_k \in \Lambda^\perp$ , where

$$\Lambda^\perp = \{\xi \in T_\lambda^*(\Gamma \times \Gamma \times \Gamma) \text{ s.t. } \lambda \in \Lambda, \langle \xi | v \rangle = 0, \forall v \in T_\lambda \Lambda\}.$$

A  $k$ -vector field  $\Sigma$  on  $\Gamma$  is said to be affine if  $[\Sigma, \overleftarrow{X}]$  is left invariant for all  $X \in \Gamma(A)$ . Here  $A$  denotes the Lie algebroid of  $\Gamma$ , and  $\overleftarrow{X}$  denotes the left invariant vector field on  $\Gamma$  corresponding to  $X$ .

The following lemma gives a useful characterization of multiplicative polyvector fields.

**Lemma 2.3** (Theorem 2.19 in [9]). *A  $k$ -vector field  $\Sigma$  is multiplicative if and only if the following three conditions hold:*

- 1)  $\Sigma$  is affine;
- 2)  $M$  is a coisotropic submanifold of  $\Gamma$ ;
- 3) for any  $\xi \in \Omega^1(M)$ ,  $\iota_{t^*(\xi)}\Sigma$  is left invariant.

**Remark 2.4.** *The statement of Theorem 2.19 in [9] contains more conditions but some of them are redundant.*

Along the base manifold  $M$ , the tangent bundle  $T\Gamma$  admits a natural decomposition

$$T\Gamma|_M = TM \oplus A,$$

where  $A$  is identified with  $T^s\Gamma|_M$ , the tangent bundle to the  $s$ -fibers along  $M$ . Denote by  $\rho : A \rightarrow TM$  the anchor map. Then  $\rho$  is equal to  $t_* : T^s\Gamma|_M \rightarrow TM$ .

Let  $Z_k$  be the set of all elements  $w$  of  $TM \wedge (\wedge^{k-1}A)$  satisfying

$$\iota_{\zeta_1} \iota_{\rho^* \zeta_2} w = -\iota_{\zeta_2} \iota_{\rho^* \zeta_1} w, \quad \forall \zeta_1, \zeta_2 \in T^*M.$$

Let  $D_\rho$  be a degree-0 derivation of  $\Gamma(\wedge^\bullet(TM \oplus A))$  such that  $D_\rho(a+b) = \rho(a)$ , for all  $a \in A$  and  $b \in TM$ .

**Lemma 2.5.** *For any  $w \in Z_k$  and  $j \geq 1$ , we have*

$$(4) \quad \iota_{\rho^* \zeta}(D_\rho^{j-1}w) = D_\rho^j(\iota_\zeta w) = \frac{1}{j+1} \iota_\zeta(D_\rho^j w), \quad \forall \zeta \in T^*M.$$

*Proof.* First, note that we have the following identities:

$$(5) \quad \iota_\zeta \circ D_\rho - D_\rho \circ \iota_\zeta = \iota_{\rho^* \zeta},$$

$$(6) \quad \iota_{\rho^* \zeta} \circ D_\rho = D_\rho \circ \iota_{\rho^* \zeta},$$

where  $\zeta \in T^*M$ , and both sides of Eqs. (5) and (6) are considered as linear maps  $\wedge^\bullet(TM \oplus A) \rightarrow \wedge^{\bullet-1}(TM \oplus A)$ .

Now we prove Eq. (4) by induction. If  $j = 1$ , the equation

$$\iota_{\rho^* \zeta} w = D_\rho(\iota_\zeta w)$$

follows from the definition of  $Z_k$ . By Eq. (5), we have

$$(\iota_\zeta \circ D_\rho - D_\rho \circ \iota_\zeta)w = \iota_{\rho^* \zeta} w = D_\rho(\iota_\zeta w).$$

It thus follows that

$$D_\rho(\iota_\zeta w) = \frac{1}{2} \iota_\zeta(D_\rho w).$$

Assume that Eq. (4) is valid for  $j \geq 1$ . Then, using Eq. (6), we have

$$\iota_{\rho^* \zeta}(D_\rho^j w) = (D_\rho \circ \iota_{\rho^* \zeta})(D_\rho^{j-1} w) = (D_\rho \circ D_\rho^j)(\iota_\zeta w) = D_\rho^{j+1}(\iota_\zeta w).$$

Moreover, using Eq. (5), we have

$$\begin{aligned} D_\rho^{j+1}(\iota_\zeta w) &= D_\rho(D_\rho^j(\iota_\zeta w)) = \frac{1}{j+1} (D_\rho \circ \iota_\zeta \circ D_\rho^j)w \\ &= \frac{1}{j+1} (\iota_\zeta \circ D_\rho - \iota_{\rho^* \zeta}) \circ D_\rho^j w = \frac{1}{j+1} \iota_\zeta(D_\rho^{j+1} w) - D_\rho^{j+1}(\iota_\zeta w), \end{aligned}$$

which implies that

$$D_\rho^{j+1}(\iota_\zeta w) = \frac{1}{j+2} \iota_\zeta(D_\rho^{j+1} w).$$

q.e.d.

**Proposition 2.6.** *Given a multiplicative  $k$ -vector field  $\Sigma$  on  $\Gamma$ , there exists a section  $\sigma \in \mathbf{\Gamma}(TM \wedge (\wedge^{k-1} A))$  such that*

$$(7) \quad \Sigma|_M = \frac{1 - e^{-D_\rho}}{D_\rho}(\sigma) = \sigma - \frac{1}{2!} D_\rho \sigma + \frac{1}{3!} D_\rho^2 \sigma + \cdots - \frac{(-1)^k}{k!} D_\rho^{k-1} \sigma.$$

Moreover,  $\sigma$  satisfies the following properties:

$$(8) \quad \begin{aligned} \partial_\Sigma(f) &= (-1)^{k-1} \iota_{df} \sigma, \quad \forall f \in C^\infty(M), \\ \iota_\zeta \iota_{\rho^* \xi} \sigma &= -\iota_\xi \iota_{\rho^* \zeta} \sigma, \quad \forall \xi, \zeta \in \Omega^1(M). \end{aligned}$$

*Proof.* Since  $M$  is coisotropic in  $\Gamma$  with respect to  $\Sigma$ , we may write

$$(9) \quad \Sigma|_M = \sigma^{1,k-1} + \sigma^{2,k-2} + \cdots + \sigma^{k,0},$$

where  $\sigma^{i,k-i} \in \mathbf{\Gamma}((\wedge^i TM) \wedge (\wedge^{k-i} A))$ .

Also observe that, for any 1-form  $\xi \in \Omega^1(M)$ , we have

$$(10) \quad \mathbf{t}^*(\xi)|_M = \xi + \rho^* \xi \in \mathbf{\Gamma}(T^*M \oplus A^*),$$

where  $T^*\Gamma|_M$  is naturally identified with  $T^*M \oplus A^*$ . On the other hand, Condition 3 of Lemma 2.3 implies that  $(\iota_{\mathbf{t}^*(\xi)}\Sigma)|_M$  is tangent to the  $\mathbf{s}$ -fibers, and therefore contains only  $\Gamma(\wedge^k A)$ -components. Using Eqs. (9) and (10), we obtain

$$(11) \quad \begin{cases} \iota_{\rho^*\xi}\sigma^{1,k-1} = -\iota_{\xi}\sigma^{2,k-2} \\ \iota_{\rho^*\xi}\sigma^{2,k-2} = -\iota_{\xi}\sigma^{3,k-3} \\ \vdots \\ \iota_{\rho^*\xi}\sigma^{k-1,1} = -\iota_{\xi}\sigma^{k,0}. \end{cases}$$

Note that, for all  $f \in C^\infty(M)$ ,

$$(12) \quad \partial_\Sigma(f) = [\Sigma, \mathbf{t}^*f]|_M = (-1)^{k+1}\iota_{\mathbf{t}^*(df)}\Sigma|_M = (-1)^{k+1}\iota_{df}\sigma^{1,k-1}.$$

Since

$$0 = \partial_\Sigma[f_1, f_2] = [\partial_\Sigma(f_1), f_2] + (-1)^{k-1}[f_1, \partial_\Sigma(f_2)],$$

it follows that

$$(13) \quad \iota_{\rho^*df_2}\iota_{df_1}\sigma^{1,k-1} + \iota_{\rho^*df_1}\iota_{df_2}\sigma^{1,k-1} = 0.$$

Let  $\sigma = \sigma^{1,k-1}$ . We will prove the following identity by induction on  $i$ :

$$(14) \quad \sigma^{i,k-i} = \frac{(-1)^{i-1}}{i!} D_\rho^{i-1} \sigma.$$

The case  $i = 1$  is obvious. Assume that Eq. (14) is valid for  $i$ . Then, by Eq. (11),

$$\begin{aligned} \iota_{\xi}\sigma^{i+1,k-i-1} &= -\iota_{\rho^*\xi}\sigma^{i,k-i} && \text{(by the induction assumption)} \\ &= \frac{(-1)^i}{i!} \iota_{\rho^*\xi} D_\rho^{i-1} \sigma && \text{(by Eq. (4))} \\ &= \frac{(-1)^i}{i!} \frac{1}{i+1} \iota_{\xi} D_\rho^i \sigma \\ &= \iota_{\xi} \left( \frac{(-1)^i}{(i+1)!} D_\rho^i \sigma \right). \end{aligned}$$

Thus Eq. (14) is proved. Therefore, Eq. (9) implies Eq. (7), Eq. (12) implies Eq. (8), and Eq. (13) implies Eq. (2.6). q.e.d.

**2.3.  $k$ -differentials of a Lie algebroid.** It is known that the Schouten bracket of two multiplicative polyvector fields on a Lie groupoid is still multiplicative. Therefore, the space of multiplicative polyvector fields is a graded Lie algebra [9]. On the level of Lie algebroids, multiplicative  $k$ -vector fields correspond to  $k$ -differentials of the Lie algebroid, whose definition we recall below.

Given a Lie algebroid  $A$ , a  $k$ -differential is a linear map

$$\partial : \Gamma(\wedge^\bullet A) \rightarrow \Gamma(\wedge^{\bullet+k-1} A)$$

satisfying

$$\partial(P \wedge Q) = (\partial P) \wedge Q + (-1)^{|P|(k-1)} P \wedge (\partial Q),$$

$$\partial[P, Q] = [\partial P, Q] + (-1)^{(|P|-1)(k-1)} [P, \partial Q],$$

for all  $P, Q \in \Gamma(\wedge^\bullet A)$ .

The commutator of a  $k_1$ -differential  $\partial_1$  and a  $k_2$ -differential  $\partial_2$  is the  $(k_1 + k_2 - 1)$ -differential

$$[\partial_1, \partial_2] = \partial_1 \circ \partial_2 - (-1)^{(k_1-1)(k_2-1)} \partial_2 \circ \partial_1.$$

**Theorem 2.7** ([9]). *Let  $\Gamma$  be a Lie groupoid, and  $A$  its Lie algebroid. Then every multiplicative  $k$ -vector field  $\Sigma$  induces a  $k$ -differential  $\partial_\Sigma$  by  $\overleftarrow{\partial}_\Sigma(P) = [\Sigma, \overleftarrow{P}]$ , for all  $P \in \Gamma(\wedge^\bullet A)$ . Here  $\overleftarrow{V}$  denotes the left invariant polyvector field on  $\Gamma$  determined by  $V$ .*

*Moreover, the map  $\Sigma \mapsto \partial_\Sigma$  is a homomorphism of graded Lie algebras, which is an isomorphism of graded Lie algebras provided  $\Gamma$  is  $s$ -connected and  $s$ -simply connected.*

In case of Lie groups,  $k$ -differentials of multiplicative  $k$ -vector fields can be described more explicitly.

**Lemma 2.8.** *Let  $\Sigma$  be a multiplicative  $k$ -vector field on a Lie group  $G$ .*

- 1) *The map  $\widehat{\sigma} : G \rightarrow \wedge^k \mathfrak{g}$  defined by  $\widehat{\sigma}(g) = L_{g^{-1}*}(\Sigma|_g)$  is a Lie group 1-cocycle.*
- 2) *The Lie algebra 1-cocycle induced by  $\widehat{\sigma}$  is the  $k$ -differential  $\partial_\Sigma$ :*

$$\left. \frac{d}{dt} \right|_{t=0} (\widehat{\sigma}|_{\exp tx}) = \left. \frac{d}{dt} \right|_{t=0} L_{\exp^{-1}tx*}(\Sigma|_{\exp tx}) = -\partial_\Sigma(x), \quad \forall x \in \mathfrak{g}.$$

**2.4. Infinitesimal data of multiplicative vector fields on Lie 2-groups.** This section is devoted to the description of infinitesimal data of multiplicative vector fields on Lie 2-groups.

**2.4.1. Multiplicative polyvector fields on Lie 2-groups.** Let  $(\Theta \xrightarrow{\Phi} G)$  be a Lie group crossed module and  $G \ltimes \Theta$  the corresponding Lie 2-group. Consider a  $k$ -vector field  $\mathbf{V} \in \mathfrak{X}^k(G \ltimes \Theta)$ .

**Definition 2.9.** *A multiplicative 0-vector field on  $G \ltimes \Theta$  is a smooth function  $f \in C^\infty(G \ltimes \Theta)$  subject to the following conditions:*

$$\begin{aligned} f(p \diamond q) &= f(p) + f(q), \quad \forall p, q \in G \ltimes \Theta; \\ f(p \star q) &= f(p) + f(q), \quad \forall p, q \in G \ltimes \Theta \text{ s.t. } \mathbf{t}(p) = \mathbf{s}(q). \end{aligned}$$

For  $k \geq 1$ , a  $k$ -vector field  $\mathbf{V}$  is called *multiplicative* if it is multiplicative with respect to both the group and the groupoid structure on  $G \ltimes \Theta$ . In other words, the graph of the group multiplication

$$\Lambda^{gp} = \{(r_1, r_2, r_1 \diamond r_2) | r_1, r_2 \in G \ltimes \Theta\}$$

and the graph of the groupoid multiplication

$$\Lambda^{gpd} = \{(r_1, r_2, r_1 \star r_2) | r_1, r_2 \in G \ltimes \Theta, \mathbf{t}(r_1) = \mathbf{s}(r_2)\}$$

are both coisotropic with respect to the  $k$ -vector field  $(\mathbf{V}, \mathbf{V}, (-1)^{k+1} \mathbf{V})$  on  $(G \ltimes \Theta) \times (G \ltimes \Theta) \times (G \ltimes \Theta)$ .

Denote the space of multiplicative  $k$ -vector fields by  $\mathfrak{X}_{\text{mult}}^k(G \ltimes \Theta)$ . The following lemma follows immediately.

**Lemma 2.10.** *When endowed with the Schouten bracket, the space of multiplicative polyvector fields*

$$\mathfrak{X}_{\text{mult}}^\bullet(G \ltimes \Theta) := \oplus_{k \geq 0} \mathfrak{X}_{\text{mult}}^k(G \ltimes \Theta)$$

*is a graded Lie algebra.*

**Remark 2.11.** *It is easy to see that  $f \in \mathfrak{X}_{\text{mult}}^0(G \ltimes \Theta)$  if and only if  $f(g, \alpha) = \nu(\alpha)$ ,  $\forall \alpha \in \Theta$  and  $g \in G$ , where  $\nu \in C^\infty(\Theta)$  satisfies  $\nu|_{\alpha\beta} = \nu|_\alpha + \nu|_\beta$  and  $\nu|_{g \triangleright \alpha} = \nu|_\alpha$ ,  $\forall \alpha, \beta \in \Theta$ .*

**2.4.2. The infinitesimal data.** It is natural to ask what is the infinitesimal data of a multiplicative  $k$ -vector field (with  $k \geq 1$ ) on a Lie 2-group. To answer this question, we need, as a first step, to describe the Lie algebroid  $A$  of the groupoid  $G \ltimes \Theta \rightrightarrows G$ .

It is simple to see that  $A$  is the transformation Lie algebroid  $G \rtimes \theta \rightarrow G$ , where the  $\theta$ -action on  $G$  is  $u \mapsto \overleftarrow{\phi}(u)$ ,  $\forall u \in \theta$ . Here the superscript  $\overleftarrow{\cdot}$  stands for the left invariant vector field on  $G$  associated to a Lie algebra element in  $\mathfrak{g}$ . It follows from Theorem 2.7 that a multiplicative  $k$ -vector field  $\mathbf{V} \in \mathfrak{X}_{\text{mult}}^k(G \ltimes \Theta)$  induces a  $k$ -differential

$$(15) \quad \partial^{\text{gpd}} : \Gamma(\wedge^\bullet A) \rightarrow \Gamma(\wedge^{\bullet+k-1} A),$$

of the Lie algebroid  $A$ . In particular, we have a map

$$\partial^{\text{gpd}} : C^\infty(G) \rightarrow \Gamma(\wedge^{k-1} A) \cong C^\infty(G, \wedge^{k-1} \theta).$$

Since  $\partial^{\text{gpd}}$  is a derivation, i.e.  $\partial^{\text{gpd}}(f_1 f_2) = f_2 \partial^{\text{gpd}}(f_1) + f_1 \partial^{\text{gpd}}(f_2)$ , for all  $f_1, f_2 \in C^\infty(G)$ ,  $\partial^{\text{gpd}}$  induces a  $\wedge^{k-1} \theta$ -valued vector field on  $G$ , which in turn can be identified with a  $\mathfrak{g} \otimes (\wedge^{k-1} \theta)$ -valued function on  $G$ . Here we identify the tangent bundle  $TG$  with  $G \times \mathfrak{g}$  by left translations. By skew symmetrization, we thus obtain a  $\mathfrak{g} \wedge (\wedge^{k-1} \theta)$ -valued function on  $G$ , denoted by  $\widehat{\delta}$ . More explicitly, we have

$$(16) \quad \partial^{\text{gpd}}(f)|_g = (-1)^{k-1} \iota_{(L_g^* df)} \widehat{\delta}|_g, \quad \forall f \in C^\infty(G), g \in G.$$

For  $k \geq 1$ , let

$$W_k = \left\{ w \in \mathfrak{g} \wedge (\wedge^{k-1}\theta) \text{ s.t. } \iota_{\zeta_1} \iota_{\phi^* \zeta_2} w = -\iota_{\zeta_2} \iota_{\phi^* \zeta_1} w, \forall \zeta_1, \zeta_2 \in \mathfrak{g}^* \right\}.$$

We have

**Lemma 2.12.** *The function  $\widehat{\delta} : G \rightarrow \mathfrak{g} \wedge (\wedge^{k-1}\theta)$  is a group 1-cocycle valued in  $W_k$ . Here  $G$  acts on  $\mathfrak{g}$  by the adjoint action, and on  $\theta$  by the induced action from the crossed module structure.*

*Proof.* Proposition 2.6 describes how a multiplicative vector field on a Lie groupoid looks like along the base manifold. Now we apply this theorem to the groupoid  $G \ltimes \Theta \rightrightarrows G$ . Identify  $TG$  with  $G \times \mathfrak{g}$  via left translations. We write  $\mathbf{V}|_g$  for the value of  $\mathbf{V}$  at  $(g, 1_\Theta)$ . For all  $g \in G$ , we have

$$\begin{aligned} (17) \quad \mathbf{V}|_g &= L_{g*} \left( \frac{1 - e^{-D_\phi}}{D_\phi} (\widehat{\delta}|_g) \right) \\ &= L_{g*} \left( \widehat{\delta}|_g - \frac{1}{2} D_\phi \widehat{\delta}|_g + \frac{1}{3!} D_\phi^2 \widehat{\delta}|_g + \cdots + \frac{(-1)^{k-1}}{k!} D_\phi^{k-1} \widehat{\delta}|_g \right). \end{aligned}$$

Here  $D_\phi : \wedge^\bullet(\mathfrak{g} \ltimes \theta) \rightarrow \wedge^\bullet(\mathfrak{g} \ltimes \theta)$  is a degree-0 derivation of the exterior algebra  $\wedge^\bullet(\mathfrak{g} \ltimes \theta)$  such that  $D_\phi(x + u) = \phi(u)$ ,  $\forall x \in \mathfrak{g}, u \in \theta$ , and, by abuse of notation,  $L_{g*}$  denotes the tangent map of the left translation by  $(g, 1_\Theta)$  on the group  $G \ltimes \Theta$ .

Since  $\mathbf{V}$  is multiplicative with respect to the group structure on  $G \ltimes \Theta$ , it follows that

$$\mathbf{V}|_{gh} = L_{g*} \mathbf{V}|_h + R_{h*} \mathbf{V}|_g,$$

where  $R_{h*}$  denotes the tangent map of the right translation by  $(h, 1_\Theta)$  in the group  $G \ltimes \Theta$ . Substituting Eq. (17) into the equation above, we see that  $\widehat{\delta}$  is indeed a Lie group 1-cocycle.

Moreover, Proposition 2.6 implies that

$$(18) \quad \iota_{\zeta_1} \iota_{\phi^* \zeta_2} (\widehat{\delta}|_g) = -\iota_{\zeta_2} \iota_{\phi^* \zeta_1} (\widehat{\delta}|_g), \quad \forall \zeta_1, \zeta_2 \in \mathfrak{g}^*.$$

As a consequence,  $\widehat{\delta}$  takes values in  $W_k$ . This concludes the proof. q.e.d.

Taking the derivative of  $\widehat{\delta}$  at the unit:

$$(19) \quad \delta(x) = - \left. \frac{d}{dt} \right|_{t=0} \widehat{\delta}|_{\exp tx}, \quad \forall x \in \mathfrak{g},$$

we obtain the following

**Corollary 2.13.** *Any multiplicative  $k$ -vector field on a Lie 2-group  $G \ltimes \Theta$  induces a Lie algebra 1-cocycle  $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge (\wedge^{k-1}\theta)$ .*

**Lemma 2.14.** *Identify  $\Theta$  with the Lie subgroup  $\{1_G\} \times \Theta$  of  $G \ltimes \Theta$ , where  $1_G$  is the unit element of  $G$ . Then any multiplicative  $k$ -vector field  $\mathbf{V}$  ( $k \geq 1$ ) is tangent to  $\Theta$ , and therefore defines a multiplicative  $k$ -vector field  $\mathbf{V}|_\Theta$  on  $\Theta$ .*

*Proof.* Let  $i$  denote the inverse map of the groupoid  $G \ltimes \Theta \rightrightarrows G$ , as described in the proof of Proposition 2.2. I.e.  $i(g, \alpha) = (g\Phi(\alpha), \alpha^{-1})$ . It is clear that  $i_*\mathbf{V} = (-1)^{k+1}\mathbf{V}$  since  $\mathbf{V}$  is multiplicative. To prove the lemma, it suffices to prove that, for any function  $f \in C^\infty(G)$ ,  $[\mathbf{V}, \mathbf{s}^*f]|_\Theta = 0$ . For all  $\alpha \in \Theta \subset G \ltimes \Theta$ , we have

$$\begin{aligned} i_*([\mathbf{V}, \mathbf{s}^*f]|_\alpha) &= (-1)^{k+1}[\mathbf{V}, \mathbf{t}^*f]|_{i(\alpha)} = (-1)^{k+1}(\overleftarrow{\partial^{\text{gpd}}(f)})|_{i(\alpha)} \\ &= (-1)^{k+1}L_{i(\alpha)*}(\partial^{\text{gpd}}(f)|_{\text{toi}(\alpha)}) = (-1)^{k+1}L_{i(\alpha)*}(\partial^{\text{gpd}}(f)|_{1_G}) = 0. \end{aligned}$$

Here  $1_G$  is the unit element of  $G$ , and  $L$  stands for the left translations with respect to the groupoid structure. The fact that  $\partial^{\text{gpd}}(f)|_{1_G} = 0$  is due to Eq. (16) and Lemma 2.12. q.e.d.

As an immediate consequence, the infinitesimal of  $\mathbf{V}|_\Theta$  gives rise to a Lie algebra 1-cocycle

$$(20) \quad \omega : \theta \rightarrow \wedge^k \theta.$$

The pair  $(\delta, \omega)$  as defined in Eqs. (19) and (20) constitutes the *infinitesimal data* of  $\mathbf{V}$ .

**2.4.3. Compatibility conditions.** This section is devoted to exploring the compatibility condition between the infinitesimal data  $\omega$  and  $\delta$ . The main theorem is the following:

**Theorem 2.15.** *Let  $(\Theta \xrightarrow{\Phi} G)$  be a crossed module of Lie groups. A multiplicative  $k$ -vector field  $\mathbf{V}$  on the Lie 2-group  $G \ltimes \Theta$  associated to this crossed module determines a pair of linear maps*

$$\begin{aligned} \omega : \theta &\rightarrow \wedge^k \theta, \\ \delta : \mathfrak{g} &\rightarrow \mathfrak{g} \wedge (\wedge^{k-1} \theta) \end{aligned}$$

*which satisfy the following three properties:*

**ID1:**  $D_\phi \omega = \delta \circ \phi$ , i.e. the diagram

$$\begin{array}{ccc} \theta & \xrightarrow{\phi} & \mathfrak{g} \\ \downarrow \omega & & \downarrow \delta \\ \wedge^k(\mathfrak{g} \ltimes \theta) & \xrightarrow{D_\phi} & \wedge^k(\mathfrak{g} \ltimes \theta) \end{array}$$

*commutes;*

**ID2:**  $\delta$  is a Lie algebra 1-cocycle valued in  $W_k$ ;



**ID3:** for all  $x \in \mathfrak{g}$  and  $u \in \theta$ ,

$$x \triangleright (\omega(u)) - \omega(x \triangleright u) = \text{pr}_{\wedge^k \theta}([u, \delta(x)]),$$

where the bracket is taken in  $\mathfrak{g} \ltimes \theta$ .

First of all, we prove that the  $k$ -differentials of  $\mathbf{V}$  with respect to both the groupoid and the group structures can be expressed in terms of the infinitesimal data  $(\omega, \delta)$ . Since  $A \cong G \times \theta$ ,  $\partial^{\text{gpd}}$  is completely determined by two  $\mathbb{R}$ -linear operators:  $\partial^{\text{gpd}} : C^\infty(G) \rightarrow C^\infty(G, \wedge^{k-1} \theta)$  and  $\partial^{\text{gpd}} : C^\infty(G, \theta) \rightarrow C^\infty(G, \wedge^k \theta)$ . The latter is determined by its value on constant functions due to the Leibniz rule.

Let  $\partial^{\text{gp}} : \mathfrak{g} \ltimes \theta \rightarrow \wedge^k(\mathfrak{g} \ltimes \theta)$  be the  $k$ -differential with respect to the group structure on  $G \ltimes \Theta$ .

**Proposition 2.16.** *The map  $\omega : \theta \rightarrow \wedge^k \theta$  satisfies*

$$(21) \quad \omega(u) = \partial^{\text{gp}}(u) = \partial^{\text{gpd}}(u), \quad \forall u \in \theta.$$

*Proof.* Every  $u \in \theta \subset \mathfrak{g} \ltimes \theta$  (considered as a constant section of the Lie algebroid  $A \cong G \times \theta$ ) determines two vector fields on  $G \ltimes \Theta$ : a vector field  $\overleftarrow{u}^{\text{gp}}$  invariant under left translations relatively to the group structure and a vector field  $\overleftarrow{u}$  invariant under left translations relatively to the groupoid structure  $G \ltimes \Theta \rightrightarrows G$ . It is simple to see that

$$(22) \quad \overleftarrow{u}^{\text{gp}} = \overleftarrow{u}.$$

Therefore,

$$\overleftarrow{\partial^{\text{gp}}(u)} = \overleftarrow{\partial^{\text{gp}}(u)}^{\text{gp}} = [\mathbf{V}, \overleftarrow{u}^{\text{gp}}] = [\mathbf{V}, \overleftarrow{u}] = \overleftarrow{\partial^{\text{gpd}}(u)}.$$

By definition, we have  $\omega(u) = \partial^{\text{gp}}(u)$ . The conclusion follows. q.e.d.

**Proposition 2.17.** *The  $k$ -differential  $\partial^{\text{gp}} : \mathfrak{g} \ltimes \theta \rightarrow \wedge^k(\mathfrak{g} \ltimes \theta)$  satisfies*

$$\begin{aligned} \partial^{\text{gp}}(u) &= \omega(u), \quad \forall u \in \theta, \\ \partial^{\text{gp}}(x) &= \frac{\mathbf{1} - e^{-D_\phi}}{D_\phi}(\delta(x)), \quad \forall x \in \mathfrak{g}. \end{aligned}$$

*Proof.* It remains to prove the second equation, which follows from a direct verification by applying Lemma 2.8 (2) and Eq. (17). q.e.d.

*Proof of Theorem 2.15.* According to Lemma 2.12, **ID2** holds. It suffices to prove **ID1** and **ID3**.

Consider the  $k$ -differential  $\partial^{\text{gpd}} : \Gamma(\wedge^\bullet A) \rightarrow \Gamma(\wedge^{\bullet+k-1} A)$  induced by  $\mathbf{V}$ . For any  $u \in \theta$ ,  $f \in C^\infty(G)$ , we have

$$(23) \quad \partial^{\text{gpd}}[u, f] = [\partial^{\text{gpd}}(u), f] + [u, \partial^{\text{gpd}}(f)].$$

Next we evaluate both sides of Eq. (23) at  $e = \mathbf{1}_G$ . Since  $\widehat{\delta}$  is a Lie group 1-cocycle according to Lemma 2.12, we have  $\widehat{\delta}|_e = 0$ . It thus follows from Eq. (16) that  $\partial^{\text{gpd}}[u, f]|_e = 0$ . On the other hand, we have

$$\begin{aligned} [\partial^{\text{gpd}}(u), f]|_e &= [\omega(u), f]|_e = (-1)^{k-1} \iota_{\phi^*(df)} \omega(u) \\ &= (-1)^{k-1} \iota_{df} ((D_{\phi} \circ \omega)(u)). \end{aligned}$$

Here we have used Proposition 2.16 and the equality  $\iota_{\phi^* \xi} = \iota_{\xi} \circ D_{\phi}$  in  $\text{Hom}(\wedge^k \theta, \wedge^{k-1} \theta)$  valid for all  $\xi \in \mathfrak{g}^*$ . Moreover, Eq. (16) implies that  $\partial^{\text{gpd}}(f)|_e = 0$ . Since the Lie algebroid  $A$  is the transformation Lie algebroid  $G \rtimes \theta$ , we have

$$\begin{aligned} [u, \partial^{\text{gpd}}(f)]|_e &= \left. \frac{d}{dt} \right|_{t=0} (\partial^{\text{gpd}}(f)|_{\exp t\phi(u)}) \\ &= (-1)^{k-1} \left. \frac{d}{dt} \right|_{t=0} \left( (L_{\exp t\phi(u)}^*(df)) \lrcorner \widehat{\delta}|_{\exp t\phi(u)} \right) \\ &= -(-1)^{k-1} \iota_{df} \delta(\phi(u)). \end{aligned}$$

Here  $\partial^{\text{gpd}}(f)$  is considered as a  $(\wedge^{k-1} \theta)$ -valued function on  $G$ , and the first equality follows from the Leibniz rule of the Lie algebroid axiom and the identity  $\partial^{\text{gpd}}(f)|_e = 0$ . Hence **ID1** follows immediately from Eq. (23).

On the other hand, the  $k$ -differential  $\partial^{\text{gp}}$  satisfies

$$\partial^{\text{gp}}[x, u] = [\partial^{\text{gp}}(x), u] + [x, \partial^{\text{gp}}(u)], \quad \forall x \in \mathfrak{g}, u \in \theta,$$

where the brackets stand for the Lie algebra bracket on  $\mathfrak{g} \ltimes \theta$ . Applying Proposition 2.17, and comparing the  $\wedge^k \theta$ -terms of both sides of the equation above, **ID3** follows immediately. q.e.d.

**Proposition 2.18.** *The map  $\omega : \theta \rightarrow \wedge^k \theta$  is a Lie algebra 1-cocycle, i.e.*

$$\omega[u, v] = [\omega(u), v] + [u, \omega(v)], \quad \forall u, v \in \theta.$$

*Proof.* Using **ID1** and **ID3** from Theorem 2.15, we have

$$\begin{aligned} \omega[u, v] &= \omega(\phi(u) \triangleright v) \\ &= \phi(u) \triangleright \omega(v) - \text{pr}_{\wedge^k \theta}([v, \delta(\phi(u))]) \\ &= [u, \omega(v)] - \text{pr}_{\wedge^k \theta}([v, D_{\phi}(\omega(u))]) \\ &= [u, \omega(v)] - [v, \omega(u)]. \end{aligned}$$

Here, in the last equality, we have used the identity

$$\text{pr}_{\wedge^k \theta}([v, D_{\phi}(\zeta)]) = [v, \zeta], \quad \forall \zeta \in \wedge^k \theta,$$

which can be verified by a straightforward computation. q.e.d.

Now we extend the two maps  $\omega$  and  $\delta$  to degree- $(k-1)$  derivations (which we denote by the same symbols by abuse of notation) on the exterior algebra  $\wedge^\bullet(\mathfrak{g} \ltimes \theta)$  by setting  $\omega(\mathfrak{g}) = 0$  and  $\delta(\theta) = 0$ .

**Proposition 2.19.** *Assume that  $\mathbf{V}_1$  and  $\mathbf{V}_2$  are multiplicative  $k_1$ - and  $k_2$ -vector fields on the Lie 2-group  $G \ltimes \Theta$ . Let  $(\omega_1, \delta_1)$  and  $(\omega_2, \delta_2)$  be their corresponding infinitesimals. Then, the infinitesimal  $(\omega_3, \delta_3)$  of  $\mathbf{V}_3 = [\mathbf{V}_1, \mathbf{V}_2]$  is given by the following formulae:*

$$(24) \quad \omega_3 = \omega_1 \circ \omega_2 - (-1)^{(k_1-1)(k_2-1)} \omega_2 \circ \omega_1,$$

$$(25) \quad \delta_3 = (\delta_1 + \omega_1) \circ \delta_2 - (-1)^{(k_1-1)(k_2-1)} (\delta_2 + \omega_2) \circ \delta_1.$$

*Proof.* Note that

$$\begin{aligned} \partial_{\mathbf{V}_3}^{\text{gp}} &= \partial_{[\mathbf{V}_1, \mathbf{V}_2]}^{\text{gp}} = [\partial_{\mathbf{V}_1}^{\text{gp}}, \partial_{\mathbf{V}_2}^{\text{gp}}] \\ &= \partial_{\mathbf{V}_1}^{\text{gp}} \circ \partial_{\mathbf{V}_2}^{\text{gp}} - (-1)^{(k_1-1)(k_2-1)} \partial_{\mathbf{V}_2}^{\text{gp}} \circ \partial_{\mathbf{V}_1}^{\text{gp}}. \end{aligned}$$

Hence Eqs. (24) and (25) follow immediately from Proposition 2.17. q.e.d.

By  $\mathcal{A}_k$  ( $k \geq 1$ ) we denote the space of pairs  $(\omega, \delta)$  of linear maps  $\omega : \theta \rightarrow \wedge^k \theta$  and  $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge (\wedge^{k-1} \theta)$  satisfying the three properties **ID1**, **ID2**, and **ID3** listed in Theorem 2.15. By  $\mathcal{A}_0$ , we denote the space of all pairs  $(\omega, \delta)$ , where  $\delta$  is the trivial map  $\mathfrak{g} \rightarrow 0$  and  $\omega : \theta \rightarrow \mathbb{R}$  satisfies  $\omega(x \triangleright u) = 0$ , for all  $x \in \mathfrak{g}$  and  $u \in \theta$ .

**Corollary 2.20.** *When endowed with the bracket defined by Eqs. (24) and (25), the direct sum  $\bigoplus_{k \geq 0} \mathcal{A}_k$  is a graded Lie algebra.*

## 2.5. The universal lifting theorem.

**2.5.1. Statement of the main theorems.** The converse of Theorem 2.15 holds as well.

**Theorem 2.21.** *Let  $G \ltimes \Theta$  be a Lie 2-group, where both  $G$  and  $\Theta$  are connected and simply connected. Given any  $(\omega, \delta) \in \mathcal{A}_k$ , there exists a unique multiplicative  $k$ -vector field  $\mathbf{V}$  on  $G \ltimes \Theta$ , whose infinitesimal corresponds to  $(\omega, \delta)$ .*

An immediate consequence is the following main result of the paper:

**Theorem 2.22** (universal lifting theorem). *Let  $G \ltimes \Theta$  be a Lie 2-group, where both  $G$  and  $\Theta$  are connected and simply connected Lie groups with Lie algebras  $\mathfrak{g}$  and  $\theta$ , respectively. There is a canonical isomorphism of graded Lie algebras*

$$\bigoplus_{k \geq 0} \mathfrak{X}_{\text{mult}}^k(G \ltimes \Theta) \cong \bigoplus_{k \geq 0} \mathcal{A}_k.$$

**2.5.2. From infinitesimal data to  $k$ -differentials of the Lie algebra  $\mathfrak{g} \ltimes \theta$ .** The  $k = 0$  case is obvious, so we will assume  $k \geq 1$  below. We will divide the proof of Theorem 2.21 into several steps.

First, since  $\delta$  is a Lie algebra 1-cocycle, it integrates into a Lie group 1-cocycle

$$\widehat{\delta} : G \rightarrow \mathfrak{g} \wedge (\wedge^{k-1} \theta)$$

such that

$$(26) \quad \delta(x) = - \left. \frac{d}{dt} \right|_{t=0} \widehat{\delta}|_{\exp tx}, \quad \forall x \in \mathfrak{g}.$$

Clearly the map  $\widehat{\delta}$  takes values in  $W_k$ .

Let  $\widehat{\omega} : \Theta \rightarrow \wedge^k \theta$  be the group 1-cocycle integrating  $\omega$ . As a direct consequence of property **ID1** from Theorem 2.15, we have

$$(27) \quad \widehat{\delta}|_{\Phi(\alpha)} = D_\phi(\widehat{\omega}|_\alpha), \quad \forall g \in G, \alpha \in \Theta.$$

Define a linear map  $\partial : \mathfrak{g} \ltimes \theta \rightarrow \wedge^k(\mathfrak{g} \ltimes \theta)$  by

$$(28) \quad \begin{cases} \partial(u) = \omega(u), & \forall u \in \theta; \\ \partial(x) = \frac{1 - e^{-D_\phi}}{D_\phi} \delta(x), & \forall x \in \mathfrak{g}. \end{cases}$$

**Proposition 2.23.** *The operator  $\partial$  defines a Lie algebra  $k$ -differential for the Lie algebra  $\mathfrak{g} \ltimes \theta$ .*

*Proof.* It suffices to prove that  $\partial$  is a Lie algebra 1-cocycle for the Lie algebra  $\mathfrak{g} \ltimes \theta$ . In fact, Proposition 2.18 implies that

$$\partial[u, v] = [\partial u, v] + [u, \partial v], \quad \forall u, v \in \theta.$$

On the other hand, it follows from a direct verification that

$$D_\phi[x, w] = [x, D_\phi(w)], \quad \forall x \in \mathfrak{g}, w \in \wedge^\bullet(\mathfrak{g} \ltimes \theta).$$

As a consequence, applying the operator  $\frac{1 - e^{-D_\phi}}{D_\phi}$  to both sides of the equation:

$$\delta[x, y] = [\delta x, y] + [x, \delta y],$$

we obtain

$$\partial[x, y] = [\partial x, y] + [x, \partial y], \quad \forall x, y \in \mathfrak{g}.$$

It remains to prove the identity

$$\partial[x, u] = [\partial x, u] + [x, \partial u], \quad \forall x \in \mathfrak{g}, u \in \theta.$$

Since  $\partial[x, u] - [x, \partial u] = \text{pr}_{\wedge^k \theta}([\delta(x), u])$  according to property **ID3** from Theorem 2.15, it suffices to prove that

$$[\partial x, u] = \text{pr}_{\wedge^k \theta}([\delta(x), u]).$$

Now

$$\begin{aligned} [\partial x, u] &= \left[ \sum_{i=0}^{k-1} \frac{(-1)^i}{(i+1)!} D_\phi^i(\delta(x)), u \right] \\ &= \sum_{j=0}^{k-1} \text{pr}_{\wedge^j \mathfrak{g} \wedge (\wedge^{k-j} \theta)} \left[ \sum_{i=0}^{k-1} \frac{(-1)^i}{(i+1)!} D_\phi^i(\delta(x)), u \right]. \end{aligned}$$

Using the definitions of  $\delta$  and  $D_\phi$ , we obtain the following identity:

$$\text{pr}_{\wedge^k \theta} \left[ \sum_{i=0}^{k-1} \frac{(-1)^i}{(i+1)!} D_\phi^i(\delta(x)), u \right] = \text{pr}_{\wedge^k \theta} [\delta(x), u].$$

For  $1 \leq j \leq k-1$ , the sum  $\text{pr}_{\wedge^j \mathfrak{g} \wedge (\wedge^{k-j} \theta)} \left[ \sum_{i=0}^{k-1} \frac{(-1)^i}{(i+1)!} D_\phi^i(\delta(x)), u \right]$  contains only the two terms

$$\text{pr}_{\wedge^j \mathfrak{g} \wedge (\wedge^{k-j} \theta)} \left( \left[ \frac{(-1)^{j-1}}{j!} D_\phi^{j-1}(\delta(x)), u \right] + \left[ \frac{(-1)^j}{(j+1)!} D_\phi^j(\delta(x)), u \right] \right)$$

and thus reduces to

$$(29) \quad \frac{(-1)^{j-1}}{j!} \text{pr}_{\wedge^j \mathfrak{g} \wedge (\wedge^{k-j} \theta)} \left( \left[ D_\phi^{j-1}(\delta(x)), u \right] - \frac{1}{j+1} \left[ D_\phi^j(\delta(x)), u \right] \right).$$

To prove that it vanishes, we need a couple of lemmas.

**Lemma 2.24.** *For any  $v \in \wedge^{k-1} \theta$  and  $1 \leq l \leq k-1$ , we have*

$$\text{pr}_{\wedge^{l-1} \mathfrak{g} \wedge (\wedge^{k-l} \theta)} \left( \left[ l D_\phi^{l-1}(v) - D_\phi^l(v), u \right] \right) = 0, \quad \forall u \in \theta.$$

*Proof.* This follows from a straightforward computation, which is left to the reader. q.e.d.

From Lemma 2.5, it follows that, for any  $w \in W_k$  and  $j \geq 1$ , we have

$$(30) \quad \iota_{\phi^* \zeta} (D_\phi^{j-1} w) = D_\phi^j (\iota_\zeta w) = \frac{1}{j+1} \iota_\zeta (D_\phi^j w), \quad \forall \zeta \in \mathfrak{g}^*.$$

Now we return to the proof of Proposition 2.23. It remains to prove that (29) vanishes. Indeed, for any  $\zeta \in \mathfrak{g}^*$ , we have

$$\begin{aligned} &\iota_\zeta \text{pr}_{\wedge^j \mathfrak{g} \wedge (\wedge^{k-j} \theta)} \left( \left[ D_\phi^{j-1}(\delta(x)), u \right] - \frac{1}{j+1} \left[ D_\phi^j(\delta(x)), u \right] \right) \\ &= \text{pr}_{\wedge^{j-1} \mathfrak{g} \wedge (\wedge^{k-j} \theta)} \left( \left[ \iota_\zeta D_\phi^{j-1}(\delta(x)), u \right] - \frac{1}{j+1} \left[ \iota_\zeta D_\phi^j(\delta(x)), u \right] \right) \\ &= \text{pr}_{\wedge^{j-1} \mathfrak{g} \wedge (\wedge^{k-j} \theta)} \left( \left[ j D_\phi^{j-1}(\iota_\zeta \delta(x)) - D_\phi^j(\iota_\zeta \delta(x)), u \right] \right) = 0. \end{aligned}$$

Here in the last two steps, we have used Eq. (30) and Lemma 2.24. This concludes the proof of the proposition. q.e.d.

**2.5.3. Multiplicative with respect to the groupoid structure.** As a consequence of Proposition 2.23, we obtain a  $k$ -vector field  $\Sigma$  on  $G \ltimes \Theta$ , which is multiplicative with respect to the group structure and whose induced  $k$ -differential with respect to the group structure on  $G \ltimes \Theta$  is  $\partial$ . Now we need to prove that  $\Sigma$  is also multiplicative with respect to the groupoid structure on  $G \ltimes \Theta \rightrightarrows G$ . For this purpose, we need an explicit expression of  $\Sigma$ . Since  $\Sigma$  is multiplicative with respect to the group structure on  $G \ltimes \Theta$ , it suffices to find an explicit expression of  $\Sigma$  along the subgroups  $\{1_G\} \times \Theta$  and  $G \times \{1_\Theta\}$ , respectively. The next two lemmas are devoted to this investigation.

The following lemma is immediate.

**Lemma 2.25.** *Identify  $\Theta$  with the subgroup  $\{1_G\} \times \Theta$  of  $G \ltimes \Theta$ . Then  $\Sigma$  is tangent to  $\Theta$  and therefore induces a multiplicative  $k$ -vector field  $\Sigma|_\Theta$  on  $\Theta$ . Moreover,  $\Sigma|_\alpha = L_{\alpha*}(\hat{\omega}|_\alpha)$ , for all  $\alpha \in \Theta$ .*

Next, we have

**Lemma 2.26.** *Along the Lie subgroup  $G \cong G \times \{1_\Theta\} \subset G \times \Theta$ ,  $\Sigma$  can be explicitly expressed by the following formula:*

$$(31) \quad \Sigma|_g = L_{g*} \left( \frac{1 - e^{-D_\phi}}{D_\phi} (\hat{\delta}|_g) \right), \quad \forall g \in G.$$

Moreover, for any  $\zeta \in \mathfrak{g}^*$ , we have

$$(32) \quad \iota_{(L_{g^{-1}}^* \zeta + \phi^* \zeta)}(\Sigma|_g) = \iota_\zeta(\hat{\delta}|_g).$$

Here  $L_{g^{-1}}^* \zeta \in T_g^* G$  and  $\phi^* \zeta \in \theta^* = T_{1_\Theta}^* \Theta$ .

*Proof.* Eq. (31) follows from integrating  $\partial(x)$  in Eq. (28). To prove Eq. (32), according to Eq. (30), we have

$$\iota_{\phi^* \zeta} \left( \frac{(-1)^{j-1}}{j!} D_\phi^{j-1} (\hat{\delta}|_g) \right) = \frac{(-1)^{j-1}}{j!} \frac{1}{j+1} \iota_\zeta D_\phi^j (\hat{\delta}|_g) = -\iota_\zeta \left( \frac{(-1)^j}{(j+1)!} D_\phi^j (\hat{\delta}|_g) \right).$$

The conclusion thus follows immediately by using Eq. (31). q.e.d.

**Proposition 2.27.** *The  $k$ -vector field  $\Sigma$  is also multiplicative with respect to the groupoid structure on  $G \ltimes \Theta \rightrightarrows G$ .*

*Proof.* We divide the proof into three steps.

(1) *The base manifold  $G$  is coisotropic with respect to  $\Sigma$ .*

For every  $g \in G$ , we have  $T_{(g, 1_\Theta)}(G \ltimes \Theta) \cong T_g G \oplus \theta$ . The conormal space of  $T_g G$  can thus be canonically identified with  $\theta^*$ . It follows that  $G$  is coisotropic with respect to  $\Sigma$  since  $\Sigma|_g$  does not contain any  $(\wedge^k \theta)$ -components according to Lemma 2.26.

(2) *For every  $\xi \in \Omega^1(G)$ ,  $\iota_{\mathbf{t}^*(\xi)} \Sigma$  is left-invariant with respect to the groupoid structure.*

For every  $(g, \alpha) \in G \ltimes \Theta$ , we identify  $T_{(g, \alpha)}(G \ltimes \Theta)$  with  $T_g G \oplus T_\alpha \Theta$ . Since  $\widehat{\omega}|_\alpha$  takes values in  $\wedge^k \theta$ , we have

$$\Sigma|_{(g, \alpha)} = \Sigma|_{g \diamond \alpha} = L_{g*}(\Sigma|_\alpha) + R_{\alpha*}(\Sigma|_g) = L_{g*}L_{\alpha*}(\widehat{\omega}|_\alpha) + R_{\alpha*}(\Sigma|_g).$$

Let  $m$  be the point  $\mathbf{t}(g, \alpha) = g\Phi(\alpha)$  of  $G$ . Choose a  $\zeta \in \mathfrak{g}^*$  and set  $\xi|_m = L_{m^{-1}}^* \zeta \in T_m^* G$ . We have, for all  $u \in \theta$ ,

$$(33) \quad \iota_{\mathbf{t}^*(\xi)} L_{g*} L_{\alpha*} u = \iota_{\phi^*(\zeta)} u,$$

which follows from the identity

$$(\mathbf{t} \circ L_g \circ L_\alpha)(\mathbf{1}_G, \beta) = g\Phi(\alpha)\Phi(\beta) = (L_{g\Phi(\alpha)} \circ \Phi)(\beta), \quad \forall \beta \in \Theta.$$

Also note that, for all  $V \in T_{(g, \mathbf{1}_\Theta)}(G \ltimes \Theta)$ ,

$$(34) \quad \iota_{\mathbf{t}^*(\xi)} R_{\alpha*} V = \iota_{(L_{g^{-1}}^* \text{Ad}_{\Phi(\alpha)}^* \zeta)} V + \iota_{(\phi^* \text{Ad}_{\Phi(\alpha)}^* \zeta)} V.$$

To prove this identity, we observe that

$$(\mathbf{t} \circ R_\alpha \circ L_g)(h, \beta) = gh\Phi(\beta)\Phi(\alpha), \quad \forall h \in G, \beta \in \Theta,$$

which implies that

$$(\mathbf{t}_* \circ R_{\alpha*} \circ L_{g*})(x, u) = (L_{g\Phi(\alpha)*} \circ \text{Ad}_{\Phi(\alpha^{-1})})(x + \phi(u)), \quad \forall x \in \mathfrak{g}, u \in \theta.$$

Thus Eq. (34) follows from a straightforward verification.

Applying Eq. (33), we obtain

$$\begin{aligned} \iota_{\mathbf{t}^*(\xi)} L_{g*}(\Sigma|_\alpha) &= \iota_{\mathbf{t}^*(\xi)} (L_{g*} L_{\alpha*}(\widehat{\omega}|_\alpha)) \\ &= L_{g*} L_{\alpha*}(\iota_{\phi^* \zeta} \widehat{\omega}|_\alpha) \\ &= L_{g*} L_{\alpha*}(\iota_\zeta D_\phi \widehat{\omega}|_\alpha) \quad (\text{by Eq. (27)}) \\ &= L_{g*} L_{\alpha*}(\iota_\zeta \widehat{\delta}|_{\Phi(\alpha)}). \end{aligned}$$

Using Eq. (34) and Lemma 2.26), we have

$$\begin{aligned} \iota_{\mathbf{t}^*(\xi)} R_{\alpha*}(\Sigma|_g) &= R_{\alpha*} \left( \iota_{L_{g^{-1}}^* (\text{Ad}_{\Phi(\alpha)}^* \zeta)} (\Sigma|_g) + \iota_{\phi^* \text{Ad}_{\Phi(\alpha)}^* \zeta} (\Sigma|_g) \right) \\ &= L_{g*} R_{\alpha*}(\iota_{\text{Ad}_{\Phi(\alpha)}^* \zeta} \widehat{\delta}|_g). \end{aligned}$$

Therefore we have

$$\begin{aligned} (\iota_{\mathbf{t}^*(\xi)} \Sigma)|_{(g, \alpha)} &= \iota_{\mathbf{t}^*(\xi)} L_{g*}(\Sigma|_\alpha) + \iota_{\mathbf{t}^*(\xi)} R_{\alpha*}(\Sigma|_g) \\ &= L_{g*} L_{\alpha*}(\iota_\zeta \widehat{\delta}|_{\Phi(\alpha)}) + L_{g*} R_{\alpha*}(\iota_{\text{Ad}_{\Phi(\alpha)}^* \zeta} \widehat{\delta}|_g) \\ &= L_{g*} L_{\alpha*} \left( \iota_\zeta \widehat{\delta}|_{\Phi(\alpha)} + \text{Ad}_{\alpha^{-1}}(\iota_{\text{Ad}_{\Phi(\alpha)}^* \zeta} \widehat{\delta}|_g) \right) \\ &= L_{g*} L_{\alpha*} \iota_\zeta \left( \widehat{\delta}|_{\Phi(\alpha)} + (\Phi(\alpha^{-1}))_* \widehat{\delta}|_g \right) \\ &= L_{g*} L_{\alpha*} \iota_\zeta \widehat{\delta}|_{g\Phi(\alpha)}, \end{aligned}$$

where, in the last step, we used the fact that  $\widehat{\delta}$  is a Lie group 1-cocycle. In particular, we have

$$(\iota_{\mathbf{t}^*(\xi)}\Sigma)|_{g\Phi(\alpha)} = (\iota_{\mathbf{t}^*(\xi)}\Sigma)|_{(g\Phi(\alpha), \mathbf{1}_\Theta)} = L_{g\Phi(\alpha)*\iota_\zeta\widehat{\delta}}|_{g\Phi(\alpha)}$$

and therefore

$$(\iota_{\mathbf{t}^*(\xi)}\Sigma)|_{(g,\alpha)} = L_{(g,\alpha)*}^{\text{gpd}}(\iota_{\mathbf{t}^*(\xi)}\Sigma)|_{g\Phi(\alpha)}.$$

This proves that  $\iota_{\mathbf{t}^*(\xi)}\Sigma$  is indeed left-invariant with respect to the groupoid structure.

**(3)** For every  $X \in \mathbf{\Gamma}(A)$ ,  $[\Sigma, \overleftarrow{X}]$  is left-invariant with respect to the groupoid structure.

It suffices to consider  $X = fu$ , where  $f \in C^\infty(G)$  and  $u \in \theta$  being considered as a constant section of  $A \cong G \times \theta$ . Then,

$$\begin{aligned} [\Sigma, \overleftarrow{X}] &= [\Sigma, (\mathbf{t}^*f)\overleftarrow{u}] \\ &= (\mathbf{t}^*f)[\Sigma, \overleftarrow{u}] + [\Sigma, \mathbf{t}^*f] \wedge \overleftarrow{u} && \text{(by Eq. (22))} \\ &= (\mathbf{t}^*f)[\Sigma, \overleftarrow{u}^{\text{gp}}] + (-1)^{k-1} \iota_{\mathbf{t}^*df}\Sigma \wedge \overleftarrow{u} \\ &= (\mathbf{t}^*f)\overleftarrow{\partial_\Sigma^{\text{gp}}(u)} + (-1)^{k-1} \iota_{\mathbf{t}^*df}\Sigma \wedge \overleftarrow{u} && \text{(by Eq. (22))} \\ &= (\mathbf{t}^*f)\overleftarrow{\partial_\Sigma^{\text{gp}}(u)} + (-1)^{k-1} \iota_{\mathbf{t}^*df}\Sigma \wedge \overleftarrow{u}, \end{aligned}$$

which is clearly left-invariant according to Claim **(2)**.

Finally, Claims **(1)**, **(2)**, and **(3)** imply that  $\Sigma$  is indeed multiplicative by Lemma 2.3. q.e.d.

*Proof of Theorem 2.21.* From the infinitesimal data  $(\omega, \delta)$ , we have constructed a multiplicative  $k$ -vector field  $\Sigma$  on the 2-group  $G \ltimes \Theta$ . Assume that the infinitesimal data corresponding to  $\Sigma$  is  $(\omega', \delta')$ . Proposition 2.17 implies that  $\omega'$  and  $\delta'$  can be recovered from  $\partial^{\text{gp}}$ , the  $k$ -differential of  $\Sigma$  with respect to the group structure on  $G \ltimes \Theta$ , by the following relations:

$$\begin{cases} \partial^{\text{gp}}(u) = \omega'(u), \\ \partial^{\text{gp}}(x) = \frac{\mathbf{1} - e^{-D_\phi}}{D_\phi}(\delta'(x)). \end{cases}$$

Since  $\partial$  is defined by Eq. (28) and  $\Sigma$  integrates  $\partial$ ,  $\partial^{\text{gp}}$  must coincide with  $\partial$ . Hence it follows that  $\omega' = \omega$  and  $\delta' = \delta$ .

Since both  $G$  and  $\Theta$  are connected and simply connected, so must be  $G \ltimes \Theta$ . Hence the multiplicative vector field  $\Sigma$  that integrates  $\partial$  must be unique. q.e.d.



### 3. Quasi-Poisson Lie 2-groups

Throughout this section,  $(\Theta \xrightarrow{\Phi} G)$  denotes a Lie group crossed module, and  $G \ltimes \Theta$  its associated Lie 2-group. By  $(\theta \xrightarrow{\phi} \mathfrak{g})$ , we denote its corresponding Lie algebra crossed module, and by  $\mathfrak{g} \ltimes \theta$  the semidirect product Lie algebra.

#### 3.1. Quasi-Poisson Lie 2-groups.

**Definition 3.1.** A quasi-Poisson structure on a Lie 2-group  $G \ltimes \Theta$  is a pair  $(\mathbf{\Pi}, \widehat{\eta})$ , where  $\mathbf{\Pi} \in \mathfrak{X}_{\text{mult}}^2(G \ltimes \Theta)$  is a multiplicative bivector field,  $\widehat{\eta}: G \rightarrow \wedge^3 \theta$  is a Lie group 1-cocycle such that

$$(35) \quad \frac{1}{2}[\mathbf{\Pi}, \mathbf{\Pi}] = \overleftarrow{\widehat{\eta}} - \overrightarrow{\widehat{\eta}},$$

and

$$(36) \quad [\mathbf{\Pi}, \overleftarrow{\widehat{\eta}}] = 0.$$

Here  $\widehat{\eta}$  is considered as a section in  $\Gamma(\wedge^3 A)$ . When  $\widehat{\eta}$  is zero,  $\mathbf{\Pi}$  defines a Poisson structure on  $G \ltimes \Theta$ . In this case, we say that  $(G \ltimes \Theta, \mathbf{\Pi})$  is a Poisson 2-group.

It is clear that  $G \ltimes \Theta \rightrightarrows G$  together with  $(\mathbf{\Pi}, \widehat{\eta})$  is a quasi-Poisson groupoid [9].

The main result of this section is the following:

**Theorem 3.2.** Any quasi-Poisson Lie 2-group  $(\mathbf{\Pi}, \widehat{\eta})$  on  $G \ltimes \Theta$  naturally induces a quasi-Lie 2-bialgebra.

Conversely, given a quasi-Lie 2-bialgebra  $(\theta, \mathfrak{g}, t)$  as in Definition 1.3, if both  $G$  and  $\Theta$  are connected and simply-connected Lie groups with Lie algebras  $\theta$  and  $\mathfrak{g}$ , respectively, then  $G \ltimes \Theta$  admits a quasi-Poisson Lie 2-group structure whose infinitesimal is isomorphic to the given quasi-Lie 2-bialgebra.

The proof is deferred to Section 3.3. In fact, from its proof, it is clear that exactly the same conclusion holds between Poisson Lie 2-groups and Lie 2-bialgebras. Thus, as an immediate consequence, we obtain the following analogue of a classical theorem of Drinfeld in the context of 2-groups.

**Corollary 3.3.** 1) There is a one-to-one correspondence between connected and simply connected quasi-Poisson Lie 2-groups and quasi-Lie 2-bialgebras.

2) There is a one-to-one correspondence between connected and simply-connected Poisson Lie 2-groups and Lie 2-bialgebras.

### 3.2. Multiplicative $k$ -vector fields generated by group 1-cocycles.

**Lemma 3.4.** *For any  $u \in \theta$ , we have*

$$(37) \quad (\text{Ad}_{(h,\beta)_\diamond}^{-1} \circ (\text{id}_\theta - \phi))(u) = ((\text{id}_\theta - \phi) \circ h_*^{-1})(u),$$

$$(38) \quad \text{Ad}_{(h,\beta)_\diamond}^{-1}(u) = (\text{Ad}_{\beta^{-1}} \circ h_*^{-1})(u) = (h\Phi(\beta))_*^{-1}(u).$$

Here  $\text{id}_\theta$  denotes the identity map on  $\theta$ .

*Proof.* A straightforward computation yields that, for all  $(g, \alpha) \in G \ltimes \Theta$ ,

$$\begin{aligned} \text{Ad}_{(h,\beta)_\diamond}^{-1}(g, \alpha) &= (h, \beta)_\diamond^{-1} \diamond (g, \alpha) \diamond (h, \beta) \\ &= \left( h^{-1}gh, ((h^{-1}g^{-1}h) \triangleright \beta^{-1})(h^{-1} \triangleright \alpha)\beta \right) \\ &= \left( \text{Ad}_{h^{-1}}g, ((\text{Ad}_{h^{-1}}g^{-1}) \triangleright \beta^{-1})(h^{-1} \triangleright \alpha)\beta \right). \end{aligned}$$

In particular, we have

$$\text{Ad}_{(h,\beta)_\diamond}^{-1}(\Phi(\alpha^{-1}), \alpha) = (\Phi(h^{-1} \triangleright \alpha^{-1}), h^{-1} \triangleright \alpha).$$

Eq. (37) thus follows immediately by taking the tangent map at  $\alpha = 1_\Theta$ . Similarly, we have

$$\text{Ad}_{(h,\beta)_\diamond}^{-1}(\mathbf{1}_G, \alpha) = (\mathbf{1}_G, (h\Phi(\beta))^{-1} \triangleright \alpha).$$

Eq. (38) follows by taking the tangent map at  $\alpha = 1_\Theta$ . q.e.d.

**Proposition 3.5.** *Let  $\widehat{\lambda} : G \rightarrow \wedge^l \theta$  be a Lie group 1-cocycle, and  $\lambda : \mathfrak{g} \rightarrow \wedge^l \theta$  the corresponding Lie algebra 1-cocycle.*

1) *The  $l$ -vector field*

$$\mathbf{C}_{\widehat{\lambda}} = \overleftarrow{\widehat{\lambda}} - \overrightarrow{\widehat{\lambda}}$$

*on the 2-group  $G \ltimes \Theta$  is multiplicative. Here  $\widehat{\lambda}$  is considered as a section in  $\Gamma(\wedge^l A)$ , and  $\overleftarrow{\widehat{\lambda}}$  and  $\overrightarrow{\widehat{\lambda}}$ , respectively, denote the left- and right-invariant  $l$ -vector fields on the groupoid  $G \ltimes \Theta \rightrightarrows G$ .*

2) *The infinitesimal data of  $\mathbf{C}_{\widehat{\lambda}}$  is*

$$\omega_\lambda = \lambda \circ \phi : \theta \rightarrow \wedge^l \theta;$$

$$\delta_\lambda = D_\phi \circ \lambda : \mathfrak{g} \rightarrow \mathfrak{g} \wedge (\wedge^{l-1} \theta).$$

3) *Let  $\partial^{\text{gpd}} : \Gamma(\wedge^\bullet A) \rightarrow \Gamma(\wedge^{\bullet+k-1} A)$  be the  $k$ -differential on the Lie algebroid  $A$  induced by a multiplicative  $k$ -vector field  $\mathbf{V}$  on  $G \ltimes \Theta$ . Then, the section  $\widehat{\sigma} = \partial^{\text{gpd}}(\widehat{\lambda}) \in \Gamma(\wedge^{k+l-1} A)$ , considered as a map  $G \rightarrow \wedge^{k+l-1} \theta$ , is a Lie group 1-cocycle. The corresponding Lie algebra 1-cocycle  $\sigma : \mathfrak{g} \rightarrow \wedge^{k+l-1} \theta$  is*

$$\sigma = \omega \circ \lambda - (-1)^{(k-1)(l-1)} \lambda \circ \delta,$$

*where  $(\omega, \delta)$  is the infinitesimal of  $\mathbf{V}$ .*

*Proof.* 1) It is clear that  $\mathbf{C}_{\hat{\lambda}}$  is multiplicative with respect to the groupoid structure. It suffices to show that  $\mathbf{C}_{\hat{\lambda}}$  is also multiplicative with respect to the group structure on  $G \ltimes \Theta$ . Define  $c : G \ltimes \Theta \rightarrow \wedge^l(\mathfrak{g} \ltimes \theta)$  by

$$c|_{(g,\alpha)} = L_{(g,\alpha)_{\diamond}^{-1}}(\mathbf{C}_{\hat{\lambda}}|_{(g,\alpha)}),$$

where  $L$  stands for the group left translations. It is well known that  $\mathbf{C}_{\hat{\lambda}}$  is multiplicative with respect to the group structure if and only if  $c$  is a group 1-cocycle, i.e.

$$(39) \quad c|_{(g,\alpha) \circ (h,\beta)} = c|_{(h,\beta)} + \text{Ad}_{(h,\beta)_{\diamond}^{-1}}(c|_{(g,\alpha)}).$$

Now a direct calculation yields

$$(40) \quad c|_{(g,\alpha)} = \hat{\lambda}|_{g\Phi(\alpha)} - (\text{id}_{\theta} - \phi)\hat{\lambda}|_g.$$

Here  $(\text{id}_{\theta} - \phi)$  extends naturally to a map  $\wedge^l \theta \rightarrow \wedge^l \theta$ , i.e.

$$(\text{id}_{\theta} - \phi)(u_1 \wedge u_2 \wedge \cdots \wedge u_l) = (\text{id}_{\theta} - \phi)u_1 \wedge (\text{id}_{\theta} - \phi)u_2 \wedge \cdots \wedge (\text{id}_{\theta} - \phi)u_l,$$

for all  $u_1, \dots, u_l \in \theta$ . Using Eq. (40) and the assumption that  $\hat{\lambda}$  is a 1-cocycle, we have

$$\begin{aligned} & \text{r.h.s of Eq. (39)} \\ &= \hat{\lambda}|_{h\Phi(\beta)} - (\text{id}_{\theta} - \phi)\hat{\lambda}|_h + (h\Phi(\beta))_*^{-1}(\hat{\lambda}|_{g\Phi(\alpha)}) - ((\text{id}_{\theta} - \phi) \circ h_*^{-1})(\hat{\lambda}|_g) \\ &= \hat{\lambda}|_{g\Phi(\alpha)h\Phi(\beta)} - (\text{id}_{\theta} - \phi)\hat{\lambda}|_{gh} \\ &= c|_{(gh, (h^{-1} \triangleright \alpha)\beta)} \\ &= \text{l.h.s. of Eq. (39)}. \end{aligned}$$

Thus,  $\mathbf{C}_{\hat{\lambda}}$  is indeed multiplicative with respect to the group structure.

2) Let  $\partial : \mathfrak{g} \ltimes \theta \rightarrow \wedge^l(\mathfrak{g} \ltimes \theta)$  be the  $l$ -differential induced by the multiplicative  $l$ -vector field  $\mathbf{C}_{\hat{\lambda}}$ . According to Lemma 2.8, we have

$$\partial(x + u) = - \left. \frac{d}{dt} \right|_{t=0} c|_{\exp t(x+u)}, \quad \forall x + u \in \mathfrak{g} \ltimes \theta.$$

Assume that  $(\omega_{\lambda}, \delta_{\lambda})$  is the infinitesimal data corresponding to  $\mathbf{C}_{\hat{\lambda}}$ . According to Proposition 2.17, we have

$$\omega_{\lambda}(u) = \partial(u) = - \left. \frac{d}{dt} \right|_{t=0} (\hat{\lambda}|_{\Phi(\exp tu)} - (\text{id}_{\theta} - \phi)\hat{\lambda}|_{1_G}) = (\lambda \circ \phi)(u).$$

Moreover,

$$\begin{aligned} \delta_{\lambda}(x) &= \text{pr}_{\mathfrak{g} \wedge (\wedge^{l-1} \theta)} \partial(x) \\ &= - \left. \frac{d}{dt} \right|_{t=0} \text{pr}_{\mathfrak{g} \wedge (\wedge^{l-1} \theta)} (\hat{\lambda}|_{\exp tx} - (\text{id}_{\theta} - \phi)\hat{\lambda}|_{\exp tx}) \\ &= - \left. \frac{d}{dt} \right|_{t=0} D_{\phi}(\hat{\lambda}|_{\exp tx}) \\ &= (D_{\phi} \circ \lambda)(x). \end{aligned}$$

Hence it follows that  $\delta_\lambda = D_\phi \circ \lambda$ .

**3)** We first prove the following formula:

$$(41) \quad \widehat{\sigma}|_g = \omega(\widehat{\lambda}|_g) + (-1)^{(k-1)(l-1)+1} \lambda(\widehat{\delta}_g) - \text{pr}_{\wedge^{k+l-1}\theta}[\widehat{\delta}|_g, \widehat{\lambda}|_g],$$

where  $\widehat{\delta} : G \rightarrow \mathfrak{g} \wedge (\wedge^{k-1}\theta)$  is the Lie group 1-cocycle corresponding to  $\delta$  as in Eq. (26). To prove it, assume that

$$\widehat{\lambda}|_g = \sum_i f_i(g) u_i \quad \text{and} \quad \widehat{\delta}|_g = \sum_j h_j(g) x_j \wedge w_j, \quad \forall g \in G,$$

where  $f_i, h_j \in C^\infty(G)$ ,  $u_i \in \wedge^l \theta$ ,  $x_j \in \mathfrak{g}$ , and  $w_j \in \wedge^{k-1} \theta$ . According to Eq. (16) and Proposition 2.16, we have

$$\begin{aligned} \widehat{\sigma}|_g &= \partial^{\text{gpd}}(\widehat{\lambda})|_g \\ &= \sum_i \left( f_i(g) \partial^{\text{gpd}}(u_i) + (\partial^{\text{gpd}} f_i)|_g \wedge u_i \right) \\ &= \sum_i \left( f_i(g) \omega(u_i) + (-1)^{k-1} \iota_{(L_g^* df_i)} \widehat{\delta}|_g \wedge u_i \right) \\ &= \omega(\widehat{\lambda}|_g) + (-1)^{k-1} \sum_{i,j} (\langle L_g^* df_i, x_j \rangle h_j(g) w_j \wedge u_i) \\ &= \omega(\widehat{\lambda}|_g) + (-1)^{k-1} \sum_j \left( h_j(g) w_j \wedge \frac{d}{dt} \Big|_{t=0} \widehat{\lambda}|_{g \exp tx_j} \right) \\ &= \omega(\widehat{\lambda}|_g) + (-1)^{k-1} \sum_j \left( h_j(g) w_j \wedge (-\lambda(x_j) - \text{pr}_{\wedge^{k+l-1}\theta}[x_j, \widehat{\lambda}|_g]) \right) \\ &= \text{r.h.s. of Eq. (41)}. \end{aligned}$$

Here in the second from the last equality, we used the identity

$$\widehat{\lambda}|_{g \exp tx_j} = \widehat{\lambda}|_{\exp tx_j} + (\exp tx_j)^{-1} \widehat{\lambda}|_g.$$

From Eq. (41), it follows that  $\widehat{\sigma}$  is indeed a Lie group 1-cocycle. Moreover, the induced Lie algebra 1-cocycle is

$$\begin{aligned} \sigma(x) &= - \frac{d}{dt} \Big|_{t=0} \widehat{\sigma}|_{\exp tx} \\ &= \frac{d}{dt} \Big|_{t=0} \left( (-1)^{(k-1)(l-1)} \lambda(\widehat{\delta}|_{\exp tx}) - \omega(\widehat{\lambda}|_{\exp tx}) + \text{pr}_{\wedge^{k+l-1}\theta}[\widehat{\delta}|_{\exp tx}, \widehat{\lambda}|_{\exp tx}] \right) \\ &= \omega(\lambda(x)) - (-1)^{(k-1)(l-1)} \lambda(\delta(x)). \end{aligned}$$

Here  $-\frac{d}{dt} \Big|_{t=0} \text{pr}_{\wedge^{k+l-1}\theta}[\widehat{\delta}|_{\exp tx}, \widehat{\lambda}|_{\exp tx}] = 0$ , since both  $\widehat{\delta}$  and  $\widehat{\lambda}$  are group 1-cocycles. This completes the proof. q.e.d.

**3.3. Proof of the main theorem.** The following result describes the infinitesimal data of a quasi-Poisson structure on the 2-group  $G \ltimes \Theta$ .

**Proposition 3.6.** *Let  $(G \ltimes \Theta, \mathbf{\Pi}, \hat{\eta})$  be a quasi-Poisson 2-group as in Definition 3.1. Let  $(\omega, \delta)$  be the corresponding infinitesimal of  $\mathbf{\Pi}$  and  $\eta : \mathfrak{g} \rightarrow \wedge^3 \theta$  the Lie algebra 1-cocycle induced by  $\hat{\eta}$ . Then the following identities hold:*

$$(42) \quad \omega^2 = \eta \circ \phi,$$

$$(43) \quad (\omega + \delta) \circ \delta = D_\phi \circ \eta,$$

$$(44) \quad \omega \circ \eta = \eta \circ \delta,$$

where  $\eta$  is identified with its extension to a degree-2 derivation of the exterior algebra  $\wedge^\bullet(\mathfrak{g} \ltimes \theta)$ .

*Proof.* Let  $\mathbf{C}_{\hat{\eta}} = \overleftarrow{\hat{\eta}} - \overrightarrow{\hat{\eta}}$ . According to Proposition 3.5,  $\mathbf{C}_{\hat{\eta}}$  is multiplicative. Moreover, its corresponding infinitesimal is  $(\eta \circ \phi, D_\phi \circ \eta)$ . By Proposition 2.19, the infinitesimal of  $\frac{1}{2}[\mathbf{\Pi}, \mathbf{\Pi}]$  is given by  $(\omega^2, (\omega + \delta) \circ \delta)$ . Thus Eq. (35) implies Eqs. (42) and (43). On the other hand, Eq. (36) is equivalent to  $\partial_{\mathbf{\Pi}}^{\text{spd}}(\hat{\eta}) = 0$ . By Lemma 3.5 (3), we have  $\omega \circ \eta - \eta \circ \delta = 0$ . This completes the proof. q.e.d.

Conversely, we have

**Proposition 3.7.** *Let  $G \ltimes \Theta$  be a Lie 2-group. If both Lie groups  $G$  and  $\Theta$  are connected and simply connected, every triple  $(\omega, \delta, \eta)$ , where  $(\omega, \delta)$  satisfies the conditions of Theorem 2.15, and  $\eta : \mathfrak{g} \rightarrow \wedge^3 \theta$  is a Lie algebra 1-cocycle satisfying the conditions of Proposition 3.6, can be uniquely integrated to a quasi-Poisson structure on  $G \ltimes \Theta$ .*

*Proof.* By Theorem 2.21, we obtain a multiplicative bivector field  $\mathbf{\Pi}$  on  $G \ltimes \Theta$  whose infinitesimal is  $(\omega, \delta)$ . Let  $\hat{\eta} : G \rightarrow \wedge^3 \theta$  be the Lie group 1-cocycle integrating  $\eta$ . By Proposition 3.5 (3),  $\partial_{\mathbf{\Pi}}^{\text{spd}}(\hat{\eta})$  vanishes since  $\omega \circ \eta - \eta \circ \delta = 0$ . Thus  $[\mathbf{\Pi}, \overleftarrow{\hat{\eta}}] = \overleftarrow{\partial_{\mathbf{\Pi}}^{\text{spd}}(\hat{\eta})} = 0$ . Moreover, Eqs. (42) and (43) imply Eq. (35) according to Proposition 3.5 (1 and 2). q.e.d.

Finally, we need the following

**Lemma 3.8.** *A quasi-Lie 2 bialgebra structure on a crossed module of Lie algebras  $(\theta \xrightarrow{\phi} \mathfrak{g})$  is equivalent to triples  $(\delta, \omega, \eta)$  of linear maps  $\delta : \mathfrak{g} \rightarrow W_2 \subset \mathfrak{g} \wedge \theta$ ,  $\omega : \theta \rightarrow \wedge^2 \theta$  and  $\eta : \mathfrak{g} \rightarrow \wedge^3 \theta$  that satisfy the following properties:*

- 1)  $D_\phi \circ \omega = \delta \circ \phi$ ;
- 2)  $\omega^2 = \eta \circ \phi$ ;
- 3)  $(\omega + \delta) \circ \delta = D_\phi \circ \eta$ ;
- 4)  $\omega \circ \eta = \eta \circ \delta$ ;

- 5)  $\eta$  is a Lie algebra 1-cocycle;
- 6)  $\delta$  is a Lie algebra 1-cocycle;
- 7)  $x \triangleright \omega(u) - \omega(x \triangleright u) = \text{pr}_{\wedge^k \theta}([u, \delta(x)])$ , for all  $x \in \mathfrak{g}$  and  $u \in \theta$ .

*Proof.* By Proposition 1.2, a weak Lie 2-coalgebra structure underlying  $(\theta \xrightarrow{\phi} \mathfrak{g})$  is equivalent to an element  $c = \check{\phi} + \check{\epsilon} + \check{\alpha} + \check{\eta} \in \mathcal{S}^{(-4)}$  such that  $\{c, c\} = 0$ . Here  $\phi$  and  $\check{\phi}$  are related by the equation:  $\phi(u) = \{\check{\phi}, u\}$ , for all  $u \in \theta$ . And  $(\theta \xrightarrow{\phi} \mathfrak{g})$  is a quasi-Lie 2-bialgebra if and only if  $\{o + c, o + c\} = 0$ , where  $o = \check{b} + \check{a}$  is the data defining the crossed module structure of  $(\theta \xrightarrow{\phi} \mathfrak{g})$ , as a special Lie 2-algebra with  $\check{h} = 0$ . Introduce the operators  $\delta$ ,  $\omega$ , and  $\eta$  by the following relations:

$$\begin{aligned} \langle \delta(x) | \xi \wedge \kappa \rangle &= -\{\{\{\check{\alpha}, x\}, \xi\}, \kappa\}; \\ \langle \omega(u) | \kappa_1 \wedge \kappa_2 \rangle &= \{\{\{\check{\epsilon}, u\}, \kappa_1\}, \kappa_2\}; \\ \langle \eta(x) | \kappa_1 \wedge \kappa_2 \wedge \kappa_3 \rangle &= \{\{\{\{\check{\eta}, x\}, \kappa_1\}, \kappa_2\}, \kappa_3\}. \end{aligned}$$

for all  $x \in \mathfrak{g}$ ,  $u \in \theta$ ,  $\xi \in \mathfrak{g}^*$ , and  $\kappa, \kappa_i \in \theta^*$ . Expand  $\{o + c, o + c\}$  and consider the result term by term. Immediately, we have the following:

- 1) the  $(\odot^2 \mathfrak{g}) \odot \mathfrak{g}^*$ -part is zero if and only if  $\delta$  is valued in  $W_2$ ;
- 2) the  $\theta \odot \mathfrak{g} \odot \theta^*$ -part is zero if and only if Condition 1) is satisfied;
- 3) the  $(\odot^3 \theta) \odot \theta^*$ -part is zero if and only if Condition 2) is satisfied;
- 4) the  $(\odot^2 \theta) \odot \mathfrak{g} \odot \mathfrak{g}^*$ -part is zero if and only if Condition 3) is satisfied;
- 5) the  $(\odot^4 \theta) \odot \mathfrak{g}^*$ -part is zero if and only if Condition 4) is satisfied;
- 6) the  $(\odot^2 \mathfrak{g}^*) \odot (\odot^3 \theta)$ -part is zero if and only if Condition 5) is satisfied;
- 7) the  $(\odot^2 \mathfrak{g}^*) \odot \mathfrak{g} \odot \theta$ -part is zero if and only if Condition 6) is satisfied;
- 8) the  $(\odot^2 \theta) \odot \mathfrak{g}^* \odot \theta^*$ -part is zero if and only if Condition 7) is satisfied.

This concludes the proof.

q.e.d.

*Proof of Theorem 3.2.* Lemma 3.8 implies that a quasi-Lie 2 bialgebra underlying the crossed module  $(\theta \xrightarrow{\phi} \mathfrak{g})$  is determined by the triple  $(\omega, \delta, \eta)$  that satisfies the conditions in Theorem 2.15 and Proposition 3.6. Thus Theorem 3.2 follows from Proposition 3.6 and Proposition 3.7.

q.e.d.

**3.4. Coboundary quasi-Poisson structures.** The following proposition describes a class of interesting examples of quasi-Poisson structures on a Lie 2-group.

**Proposition 3.9.** 1) Associated to any Lie group 1-cocycle  $\hat{\lambda} : G \rightarrow \wedge^2 \theta$ , there exists a quasi-Poisson structure  $(\Pi, \hat{\eta})$  on  $G \ltimes \Theta$  given

as follows:

$$(45) \quad \mathbf{\Pi} = \overleftarrow{\hat{\lambda}} - \overrightarrow{\hat{\lambda}},$$

$$(46) \quad \hat{\eta} = \frac{1}{2}[\hat{\lambda}, \hat{\lambda}].$$

In Eq. (45),  $\hat{\lambda}$  is considered as a section in  $\mathbf{\Gamma}(\wedge^2 A)$  and the bracket in Eq. (46) stands for the pointwise Schouten bracket on  $\wedge^\bullet \theta$ .

- 2) The infinitesimal  $(\omega_\lambda, \delta_\lambda, \eta_\lambda)$  of  $(\mathbf{\Pi}, \hat{\eta})$  as described by Proposition 3.6 is as follows:

$$\begin{aligned} \omega_\lambda &= \lambda \circ \phi, \\ \delta_\lambda &= D_\phi \circ \lambda, \\ \eta_\lambda &= \lambda \circ D_\phi \circ \lambda, \end{aligned}$$

where  $\lambda : \mathfrak{g} \rightarrow \wedge^2 \theta$  is the Lie algebra 1-cocycle induced by  $\hat{\lambda}$ .

*Proof.* The proof is standard, and is left to the reader. q.e.d.

In particular, any  $\mathbf{r} \in \wedge^2 \theta$  induces a Lie algebra 1-cocycle  $\lambda_{\mathbf{r}} : \mathfrak{g} \rightarrow \wedge^2 \theta$ :

$$(47) \quad \lambda_{\mathbf{r}}(x) = -x \triangleright \mathbf{r}, \quad \forall x \in \mathfrak{g}.$$

Therefore by Proposition 3.9, there exists a quasi-Poisson structure  $(\mathbf{\Pi}_{\mathbf{r}}, \hat{\eta}_{\mathbf{r}})$  on the 2-group  $G \ltimes \Theta$ . By a straightforward computation, we can describe this quasi-Poisson structure more explicitly:

$$(48) \quad \begin{aligned} (\mathbf{\Pi}_{\mathbf{r}})|_{(g, \alpha)} &= R_{g*} \Phi_* \mathbf{r} - L_{g*} \Phi_* \mathbf{r} + L_{\alpha*} g_*^{-1} \mathbf{r} - R_{\alpha*} g_*^{-1} \mathbf{r} \\ &\quad + [(L_{g*} \circ \Phi_*) \otimes (L_\alpha)] \mathbf{r} - [(R_{g*} \circ \Phi_*) \otimes (L_\alpha \circ g_*^{-1})] \mathbf{r}, \end{aligned}$$

and

$$(49) \quad (\hat{\eta}_{\mathbf{r}})|_g = \frac{1}{2}([\mathbf{r}, \mathbf{r}] - g_*^{-1}[\mathbf{r}, \mathbf{r}]).$$

The infinitesimal of  $(\mathbf{\Pi}_{\mathbf{r}}, \hat{\eta}_{\mathbf{r}})$  is given as follows:

$$(50) \quad \omega_{\mathbf{r}}(u) = [\mathbf{r}, u], \quad \forall u \in \theta;$$

$$(51) \quad \delta_{\mathbf{r}}(x) = -D_\phi(x \triangleright \mathbf{r}) = -x \triangleright (D_\phi \mathbf{r}), \quad \forall x \in \mathfrak{g};$$

$$(52) \quad \eta_{\mathbf{r}}(x) = -\frac{1}{2}x \triangleright [\mathbf{r}, \mathbf{r}], \quad \forall x \in \mathfrak{g}.$$

According to Lemma 3.8, the triple  $(\omega_{\mathbf{r}}, \delta_{\mathbf{r}}, \eta_{\mathbf{r}})$  also defines a quasi-Lie 2-bialgebra structure underlying  $(\theta \rightarrow \mathfrak{g})$ . In particular, if  $\eta_{\mathbf{r}} = 0$ , i.e.

$$(53) \quad x \triangleright [\mathbf{r}, \mathbf{r}] = 0, \quad \forall x \in \mathfrak{g},$$

we obtain a Lie 2-bialgebra.

**Definition 3.10.** An element  $\mathbf{r}$  of  $\wedge^2 \theta$  is called an *r-matrix* of a Lie algebra crossed module  $(\theta \xrightarrow{\phi} \mathfrak{g})$  if  $[\mathbf{r}, \mathbf{r}] \in \wedge^3 \theta$  is  $\mathfrak{g}$ -invariant, i.e. if Eq. (53) holds.

Similar to the Poisson group case, we have the following

**Theorem 3.11.** *Corresponding to any  $r$ -matrix  $\mathbf{r}$  as above, there is*

- 1) *a Poisson Lie 2-group structure  $\Pi_{\mathbf{r}}$  on  $G \ltimes \Theta$  such that*

$$\Pi_{\mathbf{r}} = \overleftarrow{\lambda}_{\mathbf{r}} - \overrightarrow{\lambda}_{\mathbf{r}},$$

*where  $\widehat{\lambda}_{\mathbf{r}} : G \rightarrow \wedge^2 \theta$  is given by Eq. (47) and*

- 2) *a Lie bialgebra crossed module underlying  $(\theta \xrightarrow{\phi} \mathfrak{g})$ .*

In this case, the Lie bracket on  $\theta^*$  is induced by the  $r$ -matrix  $\mathbf{r}$ :

$$\langle [\kappa_1, \kappa_2]_{\mathbf{r}} | u \rangle = \langle \kappa_1 \wedge \kappa_2 | [\mathbf{r}, u] \rangle, \quad \forall \kappa_1, \kappa_2 \in \theta^*, u \in \theta,$$

while the action of  $\theta^*$  on  $\mathfrak{g}^*$  is given by

$$\langle \kappa \triangleright \xi | x \rangle = \langle \kappa \wedge \phi^* \xi | x \triangleright \mathbf{r} \rangle,$$

for all  $\kappa \in \theta^*$ ,  $\xi \in \mathfrak{g}^*$ , and  $x \in \mathfrak{g}$ .

**Example 3.12.** *Let  $\theta = \mathfrak{gl}(2) \cong \mathbb{R} \text{ id} \oplus \mathfrak{sl}(2)$  and  $\mathfrak{g} = \mathfrak{sl}(2)$ . Then the projection  $\phi : \mathfrak{gl}(2) \rightarrow \mathfrak{sl}(2)$  is a Lie algebra crossed module. It is easy to check that any  $\mathbf{r} \in \wedge^2 \theta$  is indeed an  $r$ -matrix.*

**Example 3.13.** *Let  $\mathfrak{g}$  be a Lie algebra and  $\theta \subseteq \mathfrak{g}$  an ideal. Consider the Lie algebra crossed module  $\iota : \theta \rightarrow \mathfrak{g}$ , where  $\iota$  is the inclusion. Assume that  $\mathbf{r} \in \wedge^2 \theta$  such that  $[\mathbf{r}, \mathbf{r}] \in \wedge^3 \theta$  is  $\mathfrak{g}$ -invariant. Then  $\mathbf{r}$  is clearly an  $r$ -matrix. For example, we take  $\mathfrak{g} = \mathfrak{gl}(2)$  and  $\theta = \mathfrak{sl}(2)$ . Then any bivector in  $\wedge^2 \theta$  is indeed an  $r$ -matrix.*

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